Wave functions and delayed particle emission

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In this note we are concerned with β decays to unbound systems[1]. We attempt to write an explicit expression for the wave function of the daughter system that is produced in the decay. The result and its consequences are compared to other treatments in the literature. I have tried to include the most important definitions and identities in order to make the note self-contained, but otherwise clarification may be found section II of [1].

The external region

The wave function of the unbound system in the external region describes the relative motion of the fragments produced in the particle breakup. In general such a wave function can be written as a superposition of incoming and outgoing waves, $\mathscr{I}_c$ and $\mathscr{O}_c$. The channel subscript, c, besides specifying the orbital angular momentum and configuration of the breakup, also includes quantum numbers describing the internal state of the two fragments. The $\mathscr{I}_c$ and $\mathscr{O}_c$ waves are defined to have unit flux[2] and have the form

$$\mathscr{I}_c = (i^l Y_l^m)\frac{I_c}{v_c^{\frac{1}{2}} r_c}\psi_c \qquad \text{and} \qquad \mathscr{O}_c = (i^l Y_l^m)\frac{O_c}{v_c^{\frac{1}{2}} r_c}\psi_c, \tag{1}$$

where I_c and O_c are dependent only on the radial coordinate and ψ_c is the channel spin wave function which describes the internal degrees of freedom of the fragments. The two real, linearly independent, solutions to the radial Schrödinger equation in the external region are the regular and irregular Coulomb functions, F and G. I_c and O_c are defined in terms of the Coulomb functions as

$$I_c = (G_c - iF_c)\exp(i\omega_c) \qquad \text{and} \qquad O_c = (G_c + iF_c)\exp(-i\omega_c). \tag{2}$$

[1] Indeed, the results may be applicable for any transition to an unbound system.

[2] The exact meaning of this definition is discussed later.

In this expression ω_c is the Coulomb phase shift given by

$$\omega_c = \sigma_{\alpha l} - \sigma_{\alpha 0} = \sum_{n=1}^{l} \arctan\left(\frac{\eta_\alpha}{n}\right), \tag{3}$$

where α specifies the fragment pair and l is the orbital angular momentum of channel c. $\eta_\alpha = (Z_{\alpha 1} Z_{\alpha 2} e^2)/(\hbar v_c)$ is the usual Sommerfeld parameter. It should be noted that $\sigma_{\alpha l} = \arg\Gamma(1 + l_c + i\eta_\alpha)$ is also sometimes referred to as "the Coulomb phase shift", so one must keep the head straight in order not to get confused.

Since we are considering β-delayed particle breakup it seems reasonable to include only outgoing waves in the particle channels. We write the wave function in the channel, c, as

$$\Psi_{\text{ext},c} = N_c \mathscr{O}_c, \tag{4}$$

where N_c is a normalisation constant, which is as yet unknown. We define the surface functions, $\varphi_c = r_c^{-1} \psi_c (i^l Y_l^m)$ (not to be confused with the hard-sphere phase shift, ϕ_c), such that

$$\Psi_{\text{ext},c} = N_c \varphi_c O_c v_c^{-\frac{1}{2}}. \tag{5}$$

The surface functions form an orthonormal (and, supposedly, complete) set on the channel surface, $\mathscr{S}$.

The internal region

Consider the situation where a compound system is fed through only a single channel, e (for "entrance"). With unit incoming flux in e the wave function for the internal region of the compound system is given by eq. (IX.1.31) of [1]:

$$\Psi_e = -i\hbar^{\frac{1}{2}} \Omega_e \sum_{\lambda\mu} A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}} X_\lambda \tag{6}$$

Here, $\Omega_e = \exp[i(\omega_e - \phi_e)]$ with $\phi_e = \arctan(F_c/G_c)_{a_c}$ being the hard-sphere phase shift, $\Gamma_{\mu e} = 2P_e \gamma_{\mu e}^2$ is the partial width of the level, μ, and X_λ is an eigenfunction of the Hamiltonian in the internal region corresponding to the level, λ, and $A_{\lambda\mu}$ is the level matrix, defined by

$$(A^{-1})_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} - \sum_c (S_c + iP_c - B_c)\gamma_{\lambda c}\gamma_{\mu c}, \tag{7}$$

where the sum is over all breakup channels, including e. The quantities Ω_c and P_c are determined by the choice of channel radius, a_c, and the wave functions in the external region:

$$\Omega_c = \left(\frac{I_c}{O_c}\right)_{a_c}^{\frac{1}{2}} \quad ; \quad P_c = \left(\frac{\rho_c}{I_c O_c}\right)_{a_c} \quad ; \quad \rho_c = k_c r_c = \frac{M_c v_c r_c}{\hbar} \tag{8}$$

Barker proposed to treat β decays in the R-matrix framework by adding ingoing β "channels" to the compound system [2, 3]. For these channels the feeding amplitudes are denoted $G_{\lambda\beta}$ instead of $\Gamma_{\lambda\beta}$ in order to distinguish them from ordinary partial widths. This distinction is important as the $G_{\lambda\beta}$'s do not contribute to the total level widths. Citing from [3]: "These feeding amplitudes are real and may be energy-dependent". This postulate seems to allow setting $\Omega_\beta = 1$ or, equivalently, to define the feeding amplitudes as

$$G^{\frac{1}{2}}_{\lambda\beta} \equiv \Omega_\beta \Gamma^{\frac{1}{2}}_{\lambda\beta} = \text{ real number.} \tag{9}$$

Such an interpretation is supported by a similar definition for electromagnetic transitions in eq. (XIII.3.10) of [1]. According to [4] the reality of the β-decay feeding factors can be derived from first principles. We obtain the internal wave function of the compound system produced in β decay by making the substitution (9) in (6):

$$\Psi_{\text{int}} = -i\hbar^{\frac{1}{2}} \sum_{\lambda\mu} A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} X_\lambda. \tag{10}$$

It is worth noting that the formalism presented in [2, 3] was not developed specifically for β decays, but for sequential processes in general. β-delayed particle emission is clearly a sequential process, in which the parent nucleus emits a β particle, leaving the daughter nucleus in an unbound state. The ideas should be equally applicable to a "particle-delayed" particle emission, where for instance an α particle is emitted by the parent nucleus while producing an unbound daughter nucleus. The example of ${}^{12}\text{C}^* \rightarrow {}^8\text{Be}^* + \alpha \rightarrow 3\alpha$ immediately comes to mind. If this is true, the internal state of the unbound ${}^8\text{Be}^*$ daughter nucleus can be described by a wave function similar to (10), only with different feeding factors.

Matching at the channel surface

The total wave function is smooth and continuous everywhere, in particular on the channel surface at $r_c = a_c$ (for a clear and pedagogical discussion of the channel surface, see for instance section III of [5]). By construction the logarithmic derivative of the internal wave function is already matched to the logarithmic derivative of the Coulomb wave functions of the external region, so we need only be concerned with the value of the wave function on the channel surface. We require

$$\left(\Psi_{\text{int}}\right)_{a_c} = \left(\Psi_{\text{ext},c}\right)_{a_c}. \tag{11}$$

To proceed we multiply both sides by the channel surface function, φ_c^*, and integrate over the channel surface:

$$\int \varphi_c^* \left(\Psi_{\text{int}}\right)_{a_c} d\mathscr{S} = \int \varphi_c^* \left(\Psi_{\text{ext},c}\right)_{a_c} d\mathscr{S}. \tag{12}$$

First, we evaluate the LHS of (12):

$$\int \varphi_c^* (\Psi_{\rm int})_{a_c} d\mathscr{S} = \int \varphi_c^* \left(-i\hbar^{\frac{1}{2}} \sum_{\lambda\mu} A_{\lambda\mu} G_{\mu\beta}^{\frac{1}{2}} X_\lambda \right)_{a_c} d\mathscr{S}$$

$$= -i\hbar^{\frac{1}{2}} \sum_{\lambda\mu} A_{\lambda\mu} G_{\mu\beta}^{\frac{1}{2}} \int \varphi_c^* (X_\lambda)_{a_c} d\mathscr{S}$$

$$= -i \left(\frac{2M_c a_c}{\hbar} \right)^{\frac{1}{2}} \sum_{\lambda\mu} A_{\lambda\mu} G_{\mu\beta}^{\frac{1}{2}} \gamma_{\lambda c}. \tag{13}$$

Here, we have used the definition of the reduced width amplitude, $\gamma_{\lambda c}$, in eq. (III.4.8a) of [1]. M_c is the reduced mass of the fragments.

Next, we evaluate the RHS of (12):

$$\int \varphi_c^* (\Psi_{{\rm ext},c})_{a_c} d\mathscr{S} = \int \varphi_c^* \left(N_c \varphi_c O_c v_c^{-\frac{1}{2}} \right)_{a_c} d\mathscr{S}$$

$$= N_c (O_c)_{a_c} v_c^{-\frac{1}{2}} \int \varphi_c^* \varphi_c d\mathscr{S}$$

$$= N_c (O_c)_{a_c} v_c^{-\frac{1}{2}}, \tag{14}$$

where we have used the fact that the surface functions constitute an orthonormal set on the channel surface. From the relations in (8) it is clear that

$$(O_c)_{a_c} = \left(\frac{I_c O_c}{\rho_c} \right)_{a_c}^{\frac{1}{2}} \left(\frac{O_c}{I_c} \right)_{a_c}^{\frac{1}{2}} (\rho_c)_{a_c}^{\frac{1}{2}}$$

$$= P_c^{-\frac{1}{2}} \Omega_c^{-1} \left(\frac{M_c v_c a_c}{\hbar} \right)^{\frac{1}{2}}, \tag{15}$$

and the RHS of (12) becomes

$$\int \varphi_c^* (\Psi_{{\rm ext},c})_{a_c} d\mathscr{S} = N_c P_c^{-\frac{1}{2}} \Omega_c^{-1} \left(\frac{M_c a_c}{\hbar} \right)^{\frac{1}{2}}. \tag{16}$$

Equating (13) and (16) we find

$$N_c P_c^{-\frac{1}{2}} \Omega_c^{-1} \left(\frac{M_c a_c}{\hbar} \right)^{\frac{1}{2}} = -i \left(\frac{2M_c a_c}{\hbar} \right)^{\frac{1}{2}} \sum_{\lambda\mu} A_{\lambda\mu} G_{\mu\beta}^{\frac{1}{2}} \gamma_{\lambda c}$$

$$\Leftrightarrow \quad N_c = -i\Omega_c \sum_{\lambda\mu} A_{\lambda\mu} G_{\mu\beta}^{\frac{1}{2}} \Gamma_{\lambda c}^{\frac{1}{2}}, \tag{17}$$

and, finally, we obtain the external wave function in channel c by combining (17) with (4) or (5):

$$
\begin{aligned}
\Psi_{\mathrm{ext},c} &= -i\Omega_c \sum_{\lambda\mu} \left(A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} \Gamma^{\frac{1}{2}}_{\lambda c} \right) \mathscr{O}_c \\
&= -i\Omega_c \sum_{\lambda\mu} \left(A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} \Gamma^{\frac{1}{2}}_{\lambda c} \right) (i^l Y^m_l) \frac{O_c}{v_c^{\frac{1}{2}} r_c} \psi_c.
\end{aligned} \tag{18}
$$

Interpretation

In this section we discuss how the expression in eq. (18) can be transformed to a decay amplitude. It is stated in [1] that the $\mathscr{O}_c$ waves correspond to unit flux through any sphere centered at the origin. In order to figure out in which sense this should be understood we explicitly calculate the flux for such a wave. In eq. (2.4.16) of [6] the formula for calculating the probability flux of a general wave function, Ψ, is given:

$$
\mathbf{j} = -\left(\frac{i\hbar}{2M}\right)\left[\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right] = \left(\frac{\hbar}{M}\right)\mathrm{Im}\left(\Psi^*\nabla\Psi\right). \tag{19}
$$

From this expression we find the radial component of $\mathbf{j}$ for an $\mathscr{O}_c$ wave (the channel subscript c is suppressed in the following):

$$
\begin{aligned}
j_r &= \left(\frac{\hbar}{M}\right)\mathrm{Im}\left[\mathscr{O}^* \frac{\partial}{\partial r}\mathscr{O}\right] \\
&= \left(\frac{\hbar}{M}\right)\mathrm{Im}\left[(Y^m_l)^* \frac{O^*}{v^{\frac{1}{2}} r}\psi^* \frac{\partial}{\partial r}\left(Y^m_l \frac{O}{v^{\frac{1}{2}} r}\psi\right)\right] \\
&= \left(\frac{\hbar}{M}\right)|Y^m_l|^2|\psi|^2\frac{1}{v}\mathrm{Im}\left[\frac{O^*}{r}\frac{\partial}{\partial r}\left(\frac{O}{r}\right)\right],
\end{aligned} \tag{20}
$$

where in the last line we have used the fact that neither Y^m_l nor the channel wave function, ψ, is dependent on the radial coordinate. Next, we do the differentiation and use the explicit form of the O wave in terms of the Coulomb functions.

$$
\begin{aligned}
j_r &= \left(\frac{\hbar}{M}\right)|Y^m_l|^2|\psi|^2\frac{1}{v}\mathrm{Im}\left[\frac{O^*}{r}\left(\frac{1}{r}\frac{\partial O}{\partial r} - \frac{O}{r^2}\right)\right] \\
&= \left(\frac{\hbar}{M}\right)|Y^m_l|^2|\psi|^2\frac{1}{v}\mathrm{Im}\left[\frac{(G-iF)\exp(i\omega)}{r}\right. \\
&\qquad \left. \times\left(\frac{\left(\frac{\partial G}{\partial r} + i\frac{\partial F}{\partial r}\right)\exp(-i\omega)}{r} - \frac{(G+iF)\exp(-i\omega)}{r^2}\right)\right].
\end{aligned} \tag{21}
$$

This expression simplifies to

$$j_r = \left(\frac{\hbar}{M}\right)\left|Y_l^m\right|^2|\psi|^2\frac{1}{v}\frac{G\frac{\partial F}{\partial r} - F\frac{\partial G}{\partial r}}{r^2}. \tag{22}$$

Using $k = Mv/\hbar$ and the fact that $\partial/\partial r = k\partial/\partial\rho$ we get

$$\begin{aligned} j_r &= \left|Y_l^m\right|^2|\psi|^2\frac{G\frac{\partial F}{\partial\rho} - F\frac{\partial G}{\partial\rho}}{r^2} \\ &= \left|Y_l^m\right|^2|\psi|^2 r^{-2}, \end{aligned} \tag{23}$$

where in the last line we have also used the identity $F'G - G'F = 1$ (the prime denotes differentiation with respect to ρ).

To find the total probability of emission we integrate the flux over a sphere of some, arbitrary, radius R:

$$p = \int_R j_r dS = \int j_r(R)R^2 d\Omega = \int \left|Y_l^m\right|^2|\psi|^2 d\Omega = |\psi|^2, \tag{24}$$

which must be true since the internal state of the fragments do not depend on the relative coordinate of the fragments. From this result we conclude that the $\mathcal{O}_c$ wave only corresponds to unit probability of breakup through channel c if we not just integrate the flux over the entire solid angle, but also integrate over the internal coordinates, q, of the two fragments and assume the channel wave function to be normalised, such that

$$p = \int\int_R j_r dS dq = \int |\psi|^2 dq = 1. \tag{25}$$

This results leads us to rewrite the wave function in (18) as

$$\Psi_{\text{ext},c} = -S_{c\beta}\left(\frac{O_c}{v_c^{\frac{1}{2}} r_c}\right), \tag{26}$$

where the factor in paranthesis is an outgoing wave with unit probability of emission through channel c. The factor $S_{c\beta}$ is then

$$S_{c\beta}(E,\Omega,q) = i\Omega_c \sum_{\lambda\mu}\left(A_{\lambda\mu}G_{\mu\beta}^{\frac{1}{2}}\Gamma_{\lambda c}^{\frac{1}{2}}\right)(i^l Y_l^m)\psi_c \tag{27}$$

and we interpret this factor as the "decay probability amplitude". The definition of $S_{c\beta}$ makes it almost equivalent to an ordinary element of the scattering matrix (or, in the language of [1], the collision matrix), except for the inclusion of the angular dependence and the internal wave function of the decay fragments. Compare for instance to the result in (IX.1.32) of [1][3]

[3]The intelligent reader will realise that we just spent several pages in order to get from (IX.1.31) to (IX.1.32) of [1].

Comparison with literature

It is interesting to compare the amplitude in eq. (27) to some results stated in the literature. First we try to calculate the spectral density, $N(E)$, of the α particles emitted in a β-delayed α decay. We do this by squaring the amplitude in (27) and integrate over the unobserved variables:

$$\begin{aligned} N_c(E) &= \int \left|S_{c\beta}(E,\Omega,q)\right|^2 d\Omega dq \\ &= \left|\sum_{\lambda\mu} A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} \Gamma^{\frac{1}{2}}_{\lambda c}\right|^2 \int |\psi_c|^2 dq. \end{aligned} \tag{28}$$

If we assume ψ^2 to be normalised[4] we get

$$N_c(E) = \left|\sum_{\lambda\mu} A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} \Gamma^{\frac{1}{2}}_{\lambda c}\right|^2, \tag{29}$$

which agrees completely with eq. (26) of [3].

As already mentioned, the results of this note are not limited to β-delayed particle breakup but applies generally to delayed breakups. This is a case which is also investigated in the paper by Goulard [7], where a reaction of the type

$$A + B \to C^* \to D + G^* \to D + E + F \tag{30}$$

is treated. In eq. (11) of [7] the asymptotic behaviour of the wave function for the two products, E and F, is postulated to be

$$\Psi_{E,F} = \frac{\exp(ik''r'')}{r''} Y^{m''}_{l''}(\theta,\phi)\psi_{E,F} \frac{\Gamma^{\frac{1}{2}}_{\lambda'} \exp[i(\omega_{l''} - \phi_{l''})]}{E_{\lambda'} + \Delta_{\lambda'} - E_G - \frac{1}{2} i \Gamma_{\lambda'}}. \tag{31}$$

The notation in the above expression is such that k'', l'' and m'' are related to the breakup channel of G^*, and r'', θ and ϕ are the relative coordinates of the fragments E and F. The internal state of E and F is described by $\psi_{E,F}$. $\Gamma_{\lambda'}, E_{\lambda'}$ and $\Delta_{\lambda'}$ are level parameters for the intermediate level that is populated in G^*, and E_G is the relative energy of the fragments E and F, i.e. the "internal" energy of G^*. The level shift is defined in the usual way as $\Delta_{\lambda'} = -(S_c - B_c)\gamma^2_{\lambda' c}$. Comparing to our eq. (18) it is clear that eq. (31) is based on at least two approximations: The one-level approximation, which means that only a single level of the unbound system G^* is allowed to contribute, and the single-channel approximation, which means that G^* can only emit E, F through one channel. The latter approximation implies that $\Gamma_{\lambda'} = \Gamma_{\lambda' c}$. Further, it seems that a feeding factor of unit magnitude has been used.

[4]Speaking of the normalisation of unbound states is not trivial. This point requires further thought.

The same approximations can easily be applied to the result in eq. (18) in order to find an expression equivalent to eq. (31). In particular we have, so far, only been concerned with the wave function in a single decay channel, and so it is only really the single-level approximation, which modifies our result significantly. We get

$$\begin{aligned}\Psi_{E,F} &= -i\frac{O''}{v''^{\frac{1}{2}}r''}(i^{l''}Y_{l''}^{m''}(\theta,\phi))\psi_{E,F}\sum_{\lambda'\mu'}\left(A_{\lambda'\mu'}G^{\frac{1}{2}}_{\mu'\beta}\Gamma^{\frac{1}{2}}_{\lambda'c}\right)\Omega'' \\ &\to -i\frac{O''}{v''^{\frac{1}{2}}r''}\left(i^{l''}Y_{l''}^{m''}(\theta,\phi)\right)\psi_{E,F}\frac{\Gamma^{\frac{1}{2}}_{\lambda'}\exp[i(\omega_{l''}-\phi_{l''})]}{E_{\lambda'}+\Delta_{\lambda'}-E_G-\frac{1}{2}i\Gamma_{\lambda'}}, \end{aligned} \tag{32}$$

where, in the first line we have only rearranged and modified the previous notation slightly to make the result directly comparable to eq. (31), and in the second line the single-level approximation has been made. The result is very similar to that of [7], except for a few, minor details: There is an overall phase difference of $-i^{l''+1}$, and instead of the spherical wave $\exp(ik''r'')/r''$, we have O''/r''. Finally, the factor $v''^{-\frac{1}{2}}$ is not present in the result of [7]. It is my feeling that these differences are not very important, but this is a point which is open to discussion. It could be useful throughout such a discussion to keep in mind that the asymptotic behaviour of the O wave is

$$O \sim \exp\left[i(\rho-\eta\log 2\rho-\tfrac{1}{2}l\pi+\sigma_0)\right]. \tag{33}$$

Conclusion

We have found an explicit form of the wave function of the unbound daughter system produced in a β transition. The result is possible valid for any transition to unbound systems. We have shown that our result to a large degree is in agreement with other related results in the literature and, compared to eq. (11) in [7], our expression contains fewer approximations. Still, this note has only treated delayed breakup through a single particle channel, and the generalisation to multiple exit channels will be discussed in another note. Another interesting exercise would be to apply the formalism to processes where two unbound systems are involved in a sort of "delayed-delayed" breakup, for instance the β-delayed triple-α breakup of ^{12}C.

Bibliography

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