Sequential processes in the *R*-matrix language

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January 24, 2017

In a previous note an exact expression for the external wave function following a transition to an unbound system was obtained for the case of a single exit channel[1]. In the present note the channel concept is discussed in more detail, as well as the application of the formalism to situations with three particles in the final state.

Specification of channels and channel surfaces

We have earlier treated the concept of of channels without too much care and just labelled any channel-related quantity with the channel subscript c. It is perhaps useful to make explicit which parameters are used to distinguish channels from each other, and we recapitulate the discussion in section III of [2].

When the compound system separates into two fragments we use the greek letter α to specify the partition of nucleons together with the state of internal excitation of the (bound) fragments, i.e.

$$\alpha = \{\alpha_1 \alpha_2\} = \{A_1 Z_1 \lambda_1 A_2 Z_2 \lambda_2\}, \tag{1}$$

where we use λ to denote the internal energy levels of the fragments. Furthermore, the fragments may possess angular momentum, I_1 and I_2, with projections i_1 and i_2.[1] These spins can be combined in the representation where, instead of the individual spin projections, the total channel spin, s (projection ν) is used:

$$c = \{\alpha I_1 I_2 i_1 i_2\} \qquad \text{or} \qquad c = \{\alpha I_1 I_2 s \nu\}. \tag{2}$$

Finally, we also include the quantum numbers describing the relative motion of the fragments, l and m, in the channel specification, such that

$$c = \{\alpha I_1 I_2 s l \nu m\} \qquad \text{or} \qquad c = \{\alpha I_1 I_2 s l J M\}, \tag{3}$$

[1] I think that specifying I_1, I_2 is somehow redundant, since the internal eigenstates have a well-defined angular momentum anyways. This may also be the reason why I_1, I_2 appear in parantheses in [2].

where the JM-scheme is just an alternative representation, where s and l are coupled to form a total angular momentum of J. It is straightforward to switch between the to representations,

$$\Psi_{\alpha slJM} = \sum_{\nu m} \langle s l \nu m | JM \rangle \Psi_{\alpha s l \nu m} \tag{4}$$

If one of the fragments is emitted in an unbound state I think that the way of specifying channels, using λ_1 and λ_2, could run into trouble. Imagine that α_1 is emitted in a state of internal energy E_1, which is in an energy region that is dominated by broad, overlapping resonances of the α_1 system. Does it then make sense to say that only one of the eigenstates of α_1 is populated, or would it be more appropriate to use the internal energy and the spin, i.e. E_1 and I_1, for channel specification? Is this point also formally relevant for bound states of the fragments?

In the language of [2], the coordinate space of all nucleons is dubbed "configuration space". The region of configuration space corresponding to the separation of the compound nucleus into two fragments with $\{A_1 Z_1 A_2 Z_2\}$ is named the channel region. From the foregoing definitions it is clear that several channels, c, share the same channel of configuration space, that is for instance all those channels which correspond to the same partition, $\{A_1 Z_1 A_2 Z_2\}$, but which differ in some of the other quantum numbers $\{\lambda_1 \lambda_2 I_1 I_2 s l \nu m\}$. In the channel region of configuration space we define $\mathscr{S}_c$ to be the channel surface at $r_\alpha = a_\alpha$ beyond which the nuclear interaction between the two fragments is negligible. An element on the channel surface is

$$d\mathscr{S}_c = a_\alpha^2 d\Omega_c dq_\alpha, \tag{5}$$

where Ω_c is the angular coordinate of the relative motion and q_α are the internal coordinates of the fragments. To describe the internal states of α_1 and α_2 we introduce the channel spin wave functions, $\psi_{\alpha s \nu}$ (here written in the $s\nu$ scheme), which are normalised and orthogonal on the channel surface

$$\int \psi^*_{\alpha s \nu} \psi_{\alpha' s' \nu'} d\mathscr{S} = 4\pi a_\alpha^2 \delta_{\alpha s \nu, \alpha' s' \nu'}. \tag{6}$$

Where the integral is over all the channel surfaces, $\mathscr{S} = \sum_c \mathscr{S}_c$. Since $\psi_{\alpha s \nu}$ describe the internal states it cannot depend on the relative cordinates of the fragments, so from the above statement we conclude that the $\psi_{\alpha s \nu}$'s are also normalised with respect to the internal coordinates, i.e.

$$\int \psi^*_{\alpha s \nu} \psi_{\alpha' s' \nu'} dq = \delta_{\alpha s \nu, \alpha' s' \nu'}. \tag{7}$$

There are two reasons for the orthogonality with respect to α: When α and α' correspond to two different partitions of the nucleons the orthogonality is a consequence of the absence of spatial overlap between ψ_α and $\psi_{\alpha'}$ (I think one can even

be more specific and say that ψ_α only has an appreciable amplitude in the region of configuration space corresponding to the α channel). When α and α' correspond to the same partition, the orthogonality is a consequence of the orthogonality of the internal eigenstates of the compound system. Orthogonality with respect to s and ν is ensured by the orthogonality of the spin wave functions. According to [2] it is possible to think of any set c as specifying a unique channel, even if the channel spin wave functions overlap in configuration space. A detailed discussion of the concept of channels in configuration space can be found in section III of [3].

Multiple exit channels

In [1] an expression for the wave function in channel c following a transition to an unbound system was obtained:

$$\begin{aligned}\Psi_c &= -i\Omega_c \sum_{\lambda\mu}\left(A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta}\Gamma^{\frac{1}{2}}_{\lambda c}\right)(i^l Y_l^m)\frac{O_c}{v_c^{\frac{1}{2}} r_c}\psi_c \\ &= -i\Omega_{\alpha l} \sum_{\lambda\mu}\left(A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta}\Gamma^{\frac{1}{2}}_{\lambda(\alpha l)}\right)(i^l Y_l^m)\frac{O_{\alpha l}}{v_\alpha^{\frac{1}{2}} r_\alpha}\psi_{\alpha s\nu},\end{aligned} \tag{8}$$

where in the last line it has been made explicit which quantities depend on what parts of the channel specification. To facilitate the following analysis we also note that in general the wave function in the external region can be written

$$\Psi = \sum_c \Psi_c. \tag{9}$$

As already mentioned, the breakup channels are completely specified by $c = \{\alpha s \nu l m\}$ (we omit the explicit reference to I_1 and I_2, since these quantum numbers are implied by α).

Three-particle emission as a sequential process

Until now we have only considered unbound systems breaking into two fragments. If one of the fragments is itself unbound it can subsequently break up into two fragments, and we end up with three particles in the final state. This means that the wave function of the unbound fragment extends over all space and, strictly speaking, it is no longer possible to factor it out as $\psi_{\alpha s\nu}$ in the total wave function. If, however, we consider a sequential process in which the first emitted fragments no longer interact when the secondary breakup happens, we can choose to treat the relative motion of the two secondary fragments as an "internal" coordinate of the unbound fragment from the primary breakup. In this way the expression in eq. (8) is still valid if we let $\psi_{\alpha s\nu}$ depend on the relative motion of the two fragments of the secondary breakup.[2]

[2]It is possible that this kind of reasoning is faulty and meaningless, but in my simple mind it sounds resonable.

Before we start expanding the total wave function, we introduce some notation:

J_a, M_a: The total angular momentum quantum numbers of the primary compound system.

J_b, M_b, α_b: Quantum numbers describing the state of the secondary compound system.

I_i, i_i, α_i: Quantum numbers describing the state of the ith particle in the final state.

$c = \{\alpha s l \nu m\}$ Primary breakup channel.

$c' = \{\alpha' s' l' \nu' m'\}$ Secondary breakup channel.

First, we introduce some short-hand notation and write the total wave function of eq. (8) as

$$\Psi_{\alpha s l \nu m} = \Phi_{\alpha l m} \psi_{\alpha s \nu}, \tag{10}$$

where $\Phi_{\alpha l m}$ describes the relative motion of the primary breakup fragments and $\psi_{\alpha s \nu}$ is the channel spin wave function describing the internal state of the fragments. We now change to the JM-representation and write

$$\Psi_{\alpha s l J_a M_a} = \sum_{\nu m} \langle s l \nu m | J_a M_a \rangle \Phi_{\alpha l m} \psi_{\alpha s \nu}. \tag{11}$$

Using the same trick we uncouple the channel spin wave function into the wave functions of its constituents,

$$\psi_{\alpha s \nu} = \sum_{i_1 M_b} \langle I_1 J_b i_1 M_b | s \nu \rangle \psi_{\alpha_1 I_1 i_1} \psi_{\alpha_b J_b M_b}, \tag{12}$$

where $\psi_{\alpha_b J_b M_b}$ is the wave function of the unbound fragment. In this notation α_b describes the level, which is populated in the primary breakup and should not be confused with the α' which specifies the partition in the secondary breakup. Since α_b is also implicitly specified by α we modify the notation on order get rid of the redundant index, α_b, and to make the connection with the earlier expressions clearer:

$$\psi_{\alpha_b J_b M_b} \to \Psi^{\alpha}_{J_b M_b} = \sum_{\alpha' s' l'} \Psi^{\alpha}_{\alpha' s' l' J_b M_b} = \sum_{\alpha' s' l'} \sum_{\nu' m'} \langle s' l' \nu' m' | J_b M_b \rangle \Psi^{\alpha}_{\alpha' s' l' \nu' m'}. \tag{13}$$

Here, $\Psi^{\alpha}_{\alpha' s' l' \nu' m'}$ is a wave function of the same type as the one in eq. (8) fed through the level specified by α. Making a substitution equivalent to eq. (10) we obtain

$$\Psi^{\alpha}_{\alpha' s' l' \nu' m'} = \Phi^{\alpha}_{\alpha' l' m'} \psi_{\alpha' s' \nu'} = \Phi^{\alpha}_{\alpha' l' m'} \sum_{i_2 i_3} \langle I_2 I_3 i_2 i_3 | s' \nu' \rangle \psi_{\alpha_2 I_2 i_2} \psi_{\alpha_3 I_3 i_3}. \tag{14}$$

$\Phi^{\alpha}_{\alpha' l' m'}$ is here a wave function for the relative motion of the two fragments from the secondary breakup and the last equality is a result of the same type of angular momentum decoupling as in eq. (12).

Combination of eqs. (11) through (14) yields the rather formidable expression of the channel wave function

$$\Psi_{\alpha s l J_a M_a} = \sum_{\nu m} \sum_{i_1 M_b} \sum_{\alpha' s' l'} \sum_{\nu' m'} \sum_{i_2 i_3} \langle s\, l\, \nu m | J_a M_a \rangle \langle I_1 J_b\, i_1 M_b | s \nu \rangle \times \langle s' l' \nu' m' | J_b M_b \rangle \langle I_2 I_3 i_2 i_3 | s' \nu' \rangle \Phi_{\alpha l m} \Phi^{\alpha}_{\alpha' l' m'} \psi_{\alpha_1 I_1 i_1} \psi_{\alpha_2 I_2 i_2} \psi_{\alpha_3 I_3 i_3}. \quad (15)$$

Finally, the total wave function is obtained by summation over the possible exit channels:

$$\Psi = \sum_{\alpha s l J_a M_a} \Psi_{\alpha s l J_a M_a}. \quad (16)$$

(In case the reader doesn't bother to count, the above expression involves sixteen summation indices) It should be noted that if two or more of the particles in the final state are identical bosons (fermions), the wave function should be symmetrised (antisymmetrised) with respect to exchange of any identical pair. I am slightly confused about this symmetrisation, since one can also consider symmetrisation with respect to order of emission. This is perhaps a point which could be discussed in more detail.

Triple-α breakup of Carbon-12

The process we are considering is

$$^{12}\mathrm{C}^{*} \rightarrow \alpha_1 + {}^{8}\mathrm{Be}^{*} \rightarrow \alpha_1 + \alpha_2 + \alpha_3, \quad (17)$$

and so ^{8}Be plays the rôle of the unbound fragment from the primary breakup. In this case we have three α particles in the final state. Since the α particle is a spin-zero particle the Clebsch-Gordanerie of eq. (15) simplifies considerably. We have $I_1 = I_2 = I_3 = 0$ and therefore also $i_1 = i_2 = i_3 = 0$. This leads to

$$\begin{aligned} \langle I_2 I_3 i_2 i_3 | s' \nu' \rangle &= \delta_{0 s'} \delta_{0 \nu'} && \Rightarrow s' = \nu' = 0 \\ \langle s' l' \nu' m' | J_b M_b \rangle &= \delta_{l' J_b} \delta_{m' M_b} && \Rightarrow l' = J_b; m' = M_b \\ \langle I_1 J_b\, i_1 M_b | s \nu \rangle &= \delta_{J_b s} \delta_{M_b \nu} && \Rightarrow s = J_b; \nu = M_b. \end{aligned} \quad (18)$$

Furthermore the sum over α' only involves single term, since the α particle channel is the only open channel for the ^{8}Be breakup. As a consequence we can write the wave function of eq. (15) as

$$\Psi_{\alpha s l J_a M_a} = \sum_{M_b m} \langle J_b\, l\, M_b m | J_a M_a \rangle \Phi_{\alpha l m} \Phi^{\alpha}_{\alpha' J_b M_b} \psi_{\alpha_1} \psi_{\alpha_2} \psi_{\alpha_3}. \quad (19)$$

Due to the properties of the Clebsch-Gordan coefficients we only have contributions to the sum if $M_b + m = M_a$, and we get

$$\Psi_{\alpha s l J_a M_a} = \sum_{M_b} \langle J_b\, l\, M_b (M_a - M_b) | J_a M_a \rangle \Phi_{\alpha l (M_a - M_b)} \Phi^{\alpha}_{\alpha' J_b M_b} \psi_{\alpha_1} \psi_{\alpha_2} \psi_{\alpha_3}. \tag{20}$$

We now substitute the Φ's with the full expression and find

$$\begin{aligned} \Psi_{\alpha s l J_a M_a} = \sum_{M_b} & \langle J_b\, l\, M_b (M_a - M_b) | J_a M_a \rangle \psi_{\alpha_1} \psi_{\alpha_2} \psi_{\alpha_3} \\ & \times \left[-i\Omega_{\alpha l} \sum_{\lambda\mu} A_{\lambda\mu} G^{\frac{1}{2}}_{\mu\beta} \Gamma^{\frac{1}{2}}_{\lambda(\alpha l)} \left(i^l Y_l^{M_a - M_b} \right) \frac{O_{\alpha l}}{v_\alpha^{\frac{1}{2}} r_\alpha} \right] \\ & \times \left[-i\Omega_{\alpha' J_b} \sum_{\lambda'} A'_{\lambda'\mu'} G^{\frac{1}{2}}_{\mu'\alpha} \Gamma^{\frac{1}{2}}_{\lambda'(\alpha' J_b)} \left(i^{J_b} Y_{J_b}^{M_b} \right) \frac{O_{\alpha' J_b}}{v_{\alpha'}^{\frac{1}{2}} r_{\alpha'}} \right]. \end{aligned} \tag{21}$$

Here, A and A' denotes the level matrices of the ^{12}C and ^{8}Be systems, respectively, and $G^{1/2}_{\mu\beta}$ is the β decay feeding factor of the level μ in ^{12}C. Since α determines which level, μ', that is fed in the ^{8}Be system, it seems reasonable to set $G^{1/2}_{\mu'\alpha} = 1$ and omit the sum over μ', since this sum is done implicitly when summing over α.

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