Angular correlations

Consider an initial resonance of spin j_1 which decays by emission of a spin-0 particle (e.g. an alpha particle) to an intermediate resonance of spin j which subsequently decays by the emission of another spin-0 particle to a final state of spin j_2. Let l_1 denote the orbital angular momentum in the first decay and l_2 ditto for the second decay. The angle of emission of the second spinless particle relative to the first (as measured in the rest frame of the decaying intermediate resonance) is denoted β.
The correlation function $W(\beta)$ is then given by eq. (45) on p. 741 in Biedenharn & Rose (1953) with A_ν given by eq. (69a) on p. 746 and the b_ν's given by eq. (79) on p. 752.

$$W(\beta) = \sum_\nu b_\nu(l_1) b_\nu(l_2) A_\nu(l_1 l_2 j_1 j_2 j) P_\nu(\cos\beta) \tag{1}$$

The summing is extended over all even ν in the interval $0, \ldots, \min\{2l_1, 2l_2, 2j\}$.

$$b_\nu(l) = \frac{2l(l+1)}{2l(l+1) - \nu(\nu+1)} \tag{2}$$

$$A_\nu(l_1 l_2 j_1 j_2 j) = F_\nu(l_1 j_1 j) F_\nu(l_2 j_2 j) \tag{3}$$

where the F_ν's can be found in the table on page 3. Notice that only numerical values are given despite the F_ν's being expressible in fractions and square roots so the numbers can combine to give nice fractional numbers.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
8	$\frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
9	$\frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$
10	$\frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

Figure 1: The first 11 Legendre polynomials.

Let us take the decay of the 12.71 MeV state in ^{12}C as an example. This state is 1^+ so $j_1 = 1$. Having unnatural parity its decay goes through the 2^+ resonance in ^{8}Be, hence $j = 2$. Finally $j_2 = 0$ because the final state is an alpha particle. Conservation of angular momentum requires $l_2 = 2$ in the second decay. In the first decay we have three possibilities $l_1 = 1, 2, 3$, but only $l_1 = 2$ is compatible with parity conservation so that settles the matter. Hence the sum has to be extended over $\nu = 0, 2, 4$. The A_ν's are obtained by formula (3) with the F_ν's given in the table on the page 3. I get $A_0 = 1$, $A_2 = 0.1786$, and $A_4 = -0.7619$. Carrying out the sum in eq. (1) I get

$$\begin{aligned} W(\beta) &= 1 + 0.1786 \times 2 \times (3\cos^2\beta - 1) - 0.7619 \times \tfrac{9}{32} \times (35\cos^4\beta - 30\cos^2\beta + 3) \\ &= \ldots \\ &= \tfrac{15}{8}\sin^2(2\beta) \end{aligned}$$

which, by the way, is not normalized.

Whenever the decay proceeds through the 0^+ ground state in ^{8}Be there is no correlation. In some cases the first decay is not pure, e.g. when the 2^- state decays (necessarily through the 2^+ resonance in ^{8}Be) the orbital angular momentum can be both $l_1 = 1$ and $l_1 = 3$. Unfortunately things become somewhat complicated in this case since interference terms which depend on the phase shift occur in the correlation function. This is mentioned in Biedenharn & Rose (1953) on page 752. Without knowledge of the phase shift I do not see how we can derive the correct correlation function. One possibility would be to neglect the high-l channel because of the increased angular momentum barrier, but I do not know how good an approximation that is. Come to think of it the same problem applies to many other states in ^{12}C, e.g. the 2^+ and the 4^+ state.

TABLE I(a). $F_2(1j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
1	0.7071	−0.3536	0.0707	0	0
2	0	0.4183	−0.4183	0.1195	0
3	0	0	0.3464	−0.4330	0.1443
4	0	0	0	0.3134	−0.4387
5	0	0	0	0	0.2944

TABLE Ib. $F_2(2j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
1	0	−0.3535	0.3535	−0.1010	0
2	−0.5976	−0.2988	0.1281	0.3415	−0.1707
3	0	−0.4949	−0.1237	0.2268	0.3093
4	0	0	−0.4477	−0.0448	0.2645
5	0	0	0	−0.4206	0

TABLE I(c). $F_4(2j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
2	−1.069	0.7127	−0.3054	0.0764	−0.0085
3	0	−0.4467	0.6700	−0.4467	0.1489
4	0	0	−0.3044	0.6087	−0.4981
5	0	0	0	−0.2428	0.5665

TABLE I(d). $F_2(3j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
1	0	0	−0.4243	0.5303	−0.1768
2	0	−0.7171	−0.1793	0.3287	0.4482
3	−0.8660	−0.6495	−0.2742	0.1443	0.4330
4	0	−0.7835	−0.4701	−0.0855	0.2678
5	0	0	−0.7360	−0.3680	0.0170

TABLE I(e). $F_4(3j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
2	0	0.0891	−0.1336	0.0891	−0.0297
3	0.2132	0.0355	−0.1066	−0.0355	0.1044
4	0	0.1453	−0.0484	−0.1012	0.0132
5	0	0	0.1159	−0.0773	−0.0852

related to the ratio of reduced matrix elements by

$$\delta^2 \equiv \frac{I_{L_1+1}}{I_{L_1}} = \frac{(j_1\|L_1+1\|j)^2}{(j_1\|L_1\|j)^2}. \tag{67}$$

Very often $I_{L_1+1}/I_{L_1} \ll 1$ and only linear terms in δ need be retained, (56). Of course, while the correlation function for pure multipoles is parity independent, the case of mixed multipoles gives a parity determination only if it is assumed that a M_L, E_{L+1} mixture is much more likely than an E_L, M_{L+1} mixture (76a).

It should also be emphasized that the presence of interference in the correlation may change the correla-

TABLE I(f). $F_6(3j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
3	1.3056	−0.9792	0.5440	−0.2176	0.0593
4	0	0.4214	−0.7585	0.6895	−0.3831
5	0	0	0.2420	−0.6049	0.6979

TABLE I(g). $F_2(4j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
1	0	0	0	−0.4293	0.6010
2	0	0	−0.7257	−0.0726	0.4288
3	0	−0.8763	−0.5258	−0.0956	0.2995
4	−0.9687	−0.8234	−0.5554	−0.2101	0.1447
5	0	−0.9099	−0.6825	−0.3787	−0.0437

TABLE I(h). $F_4(4j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
2	0	0	0.1718	−0.3436	0.2811
3	0	0.4112	−0.1371	−0.2866	0.0374
4	0.6034	0.3017	−0.0901	−0.2860	−0.1408
5	0	0.4814	0.0802	−0.2239	−0.2592

TABLE I(i). $F_6(4j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
3	0	0.0218	−0.0392	0.0356	−0.0198
4	0.0674	−0.0034	−0.0346	0.0104	0.0243
5	0	0.0387	−0.0290	−0.0190	0.0242

TABLE I(j). $F_8(4j_1j)$ for integer spins.

j \ j_1	0	1	2	3	4
4	−1.4809	1.1847	−0.7539	0.3770	−0.1450
5	0	−0.3893	0.7785	−0.8384	0.5989

tion markedly and the correlation measurement is a sensitive method for the determination of mixture ratios.

Equations (64) and (65) define the standard $\gamma-\gamma$ correlation. In particular Eq. (65a) is the standard $\gamma-\gamma$ correlation for pure multipoles. We write the latter in the form

$$w(\beta) = \sum_{\nu} A_\nu P_\nu(\cos\beta) \tag{68}$$

and renormalize so that $A_0 = 1$ corresponding to unit value for the average of the correlation function w. Then

$$A_\nu = F_\nu(L_1 j_1 j) F_\nu(L_2 j_2 j), \tag{69a}$$

where

$$F_\nu(Lj_1j) = (-)^{j_1-j-1}(2j+1)^{\frac{1}{2}}(2L+1) \times C(LL\nu;1-1)W(jjLL;\nu j_1) \tag{69b}$$

so that also $F_0 = 1$.