Solid Angle of a Rectangular Plate

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The solid angle covered by a rectangular plate of length a and width b at a distance d to the observer is calculated.

I. GEOMETRY

We consider a rectangular plate of size $a \times b$ (area ab) at a distance d to an observer. We consider only the special alignment where the vector normal of the plate surface starting at the plate center points to the observer, which means the plate surface is perpendicular to the line of sight.

In a Cartesian coordinate system with the center of coordinates at the observer, the plate surface coordinates are confined to

$$-a/2 \le x \le a/2; \quad -b/2 \le y \le b/2; \quad z = d. \tag{1}$$

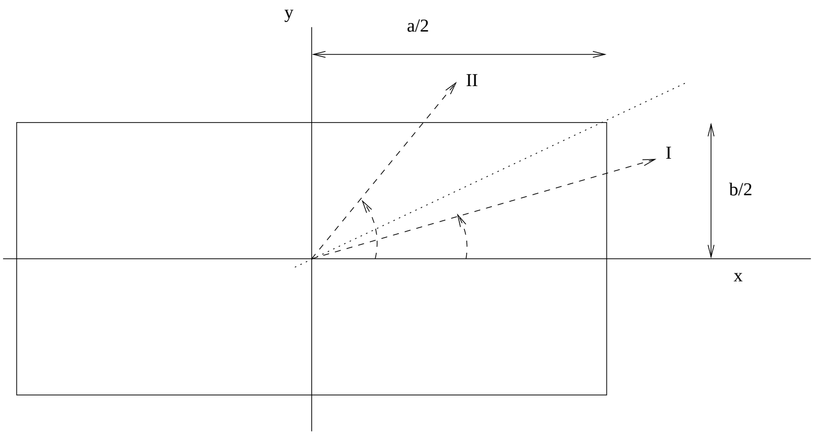

FIG. 1: The plate seen by the observer looking into the direction $+z$.

II. COORDINATE LIMITS

In a spherical coordinate system centered at the observer, with polar angle θ ($0 \le \theta \le \pi$) and azimuth angle φ ($0 \le \varphi \le 2\pi$) the solid angle of the object is

$$\Omega = \int \sin\theta \, d\theta \, d\varphi, \tag{2}$$

where the two angular coordinates scan the surface of the object. We transform the Cartesian coordinates (1) to spherical coordinates and perform the double integral, using the generic

$$x = r \sin\theta \cos\varphi, \tag{3}$$
$$y = r \sin\theta \sin\varphi, \tag{4}$$
$$z = r \cos\theta, \tag{5}$$

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and the inverse transformation

$$\varphi = \arctan \frac{y}{x}, \quad \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}. \tag{6}$$

The symmetry of the problem allows us to consider only one octant of the double hemisphere region seen by the observer, which reduces to a quadrant of the plate,

$$\Omega = 4 \int_{\substack{0\le x\le a/2\\ 0\le y\le b/2}} \sin\theta\, d\theta\, d\varphi. \tag{7}$$

This region of area $ab/4$ is split into two triangular regions along the diagonal of the plate, separated by a dotted line in Fig. 1:

- One triangular region (I) covers the interval

  $$0 \le \varphi \le \arctan(b/a). \tag{8}$$

  A point on the dashed straight line in Fig. 1 from the center of the plate up to the plate circumference has the coordinates x, $y = x\tan\varphi$, $z = d$. The line leaves the plate at $x = a/2$, $y = (a/2)\tan\varphi$, $z = d$, and therefore covers

  $$0 \le \theta \le \arccos \frac{d}{\sqrt{d^2 + \frac{a^2}{4} + \frac{a^2}{4}\tan^2\varphi}} \tag{9}$$

  according to (6).

- The complementary triangular region (II) covers

  $$\arctan(b/a) \le \varphi \le \pi/2 \tag{10}$$

  and

  $$0 \le \theta \le \arccos \frac{d}{\sqrt{d^2 + \frac{b^2}{4} + \frac{b^2}{4\tan^2\varphi}}}. \tag{11}$$

III. EVALUATION OF INTEGRALS

The previous four equations turn Eq. (2) into

$$\Omega = 4\left[\int_0^{\arctan\frac{b}{a}} d\varphi \int_0^{\arccos\frac{d}{\sqrt{d^2+\frac{a^2}{4}(1+\tan^2\varphi)}}} \sin\theta\, d\theta + \int_{\arctan\frac{b}{a}}^{\pi/2} d\varphi \int_0^{\arccos\frac{d}{\sqrt{d^2+\frac{b^2}{4}(1+\cot^2\varphi)}}} \sin\theta\, d\theta\right]. \tag{12}$$

In the θ-integrals we substitute $t = \cos\theta$, $dt = -\sin\theta d\theta$ with

$$\int_{\theta=0}^{\theta=\arccos\ldots} \sin\theta d\theta = -\int_1^{\ldots} dx = 1 - \ldots \tag{13}$$

to get

$$\Omega = 4\left[\int_0^{\arctan\frac{b}{a}} \left(1 - \frac{d}{\sqrt{d^2+\frac{a^2}{4}(1+\tan^2\varphi)}}\right) d\varphi + \int_{\arctan\frac{b}{a}}^{\pi/2} \left(1 - \frac{d}{\sqrt{d^2+\frac{b^2}{4}(1+\cot^2\varphi)}}\right) d\varphi\right]. \tag{14}$$

We integrate over the two constant parts and condense the trigonometric functions $1 + \ldots^2\varphi$ underneath the square roots,

$$\Omega = 4\left[\frac{\pi}{2} - \int_0^{\arctan\frac{b}{a}} \frac{d}{\sqrt{d^2+\frac{a^2}{4\cos^2\varphi}}} d\varphi - \int_{\arctan\frac{b}{a}}^{\pi/2} \frac{d}{\sqrt{d^2+\frac{b^2}{4\sin^2\varphi}}} d\varphi\right]. \tag{15}$$

In the second integral we substitute $\varphi\prime = \pi/2-\varphi$,

$$\Omega = 4\left[\frac{\pi}{2} - \int_0^{\arctan\frac{b}{a}} \frac{1}{\sqrt{1+\frac{a^2}{4d^2\cos^2\varphi}}} d\varphi - \int_0^{\arctan\frac{a}{b}} \frac{1}{\sqrt{1+\frac{b^2}{4d^2\cos^2\varphi\prime}}} d\varphi\prime\right]. \tag{16}$$

The two indefinite integrals of the type

$$I(\alpha,\varphi) \equiv \int \frac{d\varphi}{\sqrt{1+\alpha^2\frac{1}{\cos^2\varphi}}} = \int \frac{\cos\varphi d\varphi}{\sqrt{\cos^2\varphi+\alpha^2}} \tag{17}$$

are solved with the substitution $v = \frac{\sin\varphi}{\sqrt{1+\alpha^2}}$, $dv = d\varphi\frac{\cos\varphi}{\sqrt{1+\alpha^2}}$, $v^2(1+\alpha^2) = \sin^2\varphi = 1-\cos^2\varphi$, [1, 2.584.3]

$$I(\alpha,\varphi) = \int \frac{\sqrt{1+\alpha^2}dv}{\sqrt{1+\alpha^2-v^2(1+\alpha^2)}} = \int \frac{dv}{\sqrt{1-v^2}} = \arcsin v = \arcsin\frac{\sin\varphi}{\sqrt{1+\alpha^2}}. \tag{18}$$

Applied to (16) we have [2, 3]

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\frac{\sin\arctan\frac{b}{a}}{\sqrt{1+\frac{a^2}{4d^2}}} - \arcsin\frac{\sin\arctan\frac{a}{b}}{\sqrt{1+\frac{b^2}{4d^2}}}\right]. \tag{19}$$

The sines simplify according to the column `tan x=a` of [4, 4.3.45]

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\frac{\frac{b}{a}}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{1+\left(\frac{b}{a}\right)^2}} - \{a\leftrightarrow b\}\right] \tag{20}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\frac{b}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{a^2+b^2}} - \{a\leftrightarrow b\}\right], \tag{21}$$

where $a\leftrightarrow b$ means the previous term is to be repeated with roles of a and b interchanged. With [4, 4.4.32], we combine the two arc-sines into one,

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\left(\frac{b}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{a^2+b^2}}\sqrt{1-\frac{a^2}{[1+(\frac{b}{2d})^2][a^2+b^2]}} + \{a\leftrightarrow b\}\right)\right] \tag{22}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\left(\frac{b^2\sqrt{1+(\frac{a}{2d})^2+(\frac{b}{2d})^2}}{\sqrt{1+(\frac{a}{2d})^2}\sqrt{1+(\frac{b}{2d})^2}(a^2+b^2)} + \{a\leftrightarrow b\}\right)\right] \tag{23}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\frac{\sqrt{1+(\frac{a}{2d})^2+(\frac{b}{2d})^2}}{\sqrt{1+(\frac{a}{2d})^2}\sqrt{1+(\frac{b}{2d})^2}}\right]. \tag{24}$$

IV. RESULT

With the definition of two cone parameters $\alpha \equiv a/(2d)$ and $\beta \equiv b/(2d)$, the previous equation yields the known result [5]

$$\Omega(a,b,d) = 4\arccos\sqrt{\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)}}. \tag{25}$$

Taylor expansions for small values of α and β are

$$\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)} \approx 1-(\alpha\beta)^2+\alpha^2\beta^2(\alpha^2+\beta^2), \tag{26}$$

$$\sqrt{\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)}} \approx 1-\frac{(\alpha\beta)^2}{2}+\frac{(\alpha\beta)^2}{2}(\alpha^2+\beta^2), \tag{27}$$

$$\Omega \approx 4\alpha\beta-2\alpha\beta(\alpha^2+\beta^2)+\frac{3}{2}\alpha\beta(\alpha^4+\beta^4)+\frac{5}{3}(\alpha\beta)^3-\frac{5}{4}\alpha\beta(\alpha^6+\beta^6)-\frac{7}{4}(\alpha\beta)^3(\alpha^2+\beta^2). \tag{28}$$

The leading term $\Omega \approx 4\alpha\beta = ab/d^2$ is the ratio of the plate area over the squared distance.

Appendix A: Off-Axis Cases

The more generic case occurs when the shortest distance d from the observer to the plane which contains the plate leads to a point different from the plate center. Figure 2 shows the case when this pointing direction would not meet the plate at all but miss it by distances A and B relative to the closest edge of the plate. The line of sight hits the

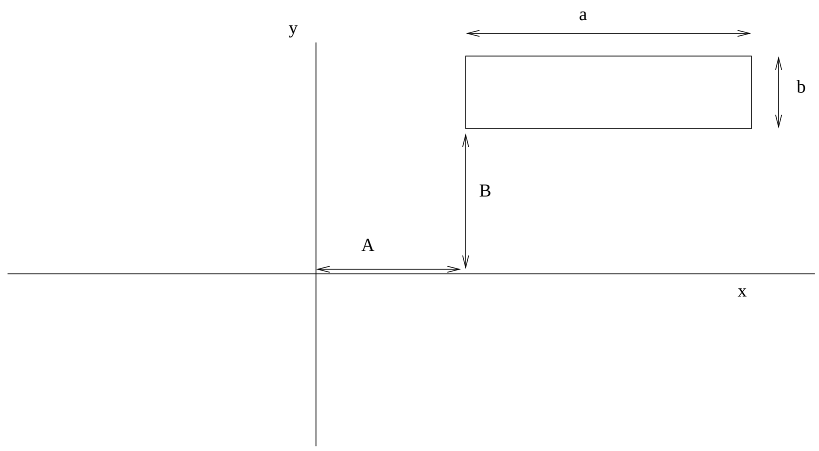

FIG. 2: Lengths in the off-axis 1-quadrant geometry, Eq. (A2).

plane of the plate at the Cartesian coordinates $(0,0,d)$; the center of the plate is at $(A+a/2, B+b/2, d)$. Therefore the angle τ between the line of sight and the direction toward the plate center is

$$\cos\tau = \frac{1}{\sqrt{1+(\frac{A+a/2}{d})^2+(\frac{B+b/2}{d})^2}}. \tag{A1}$$

Suitable symmetric decomposition of the area ab into more areas which individually meet the requirement of formula (25) for $\Omega(a,b,d)$ is shown in Figure 3. There is the virtual plate of size $2(A+a)\times 2(B+b)$ of which encompasses all the quadrangles, a horizontal middle strip of size $2(A+a)\times 2B$, a vertical middle strip of size $2A\times 2(B+b)$, and the center quadrangle of size $2A\times 2B$. Superposition of the solid angles of these with suitable corrections for multiply counted areas yields for the solid angle of the $a\times b$ rectangle in Fig. 2

$$\frac{\Omega(2(A+a),2(B+b),d)-\Omega(2A,2(B+b),d)-\Omega(2(A+a),2B,d)+\Omega(2A,2B,d)}{4}. \tag{A2}$$

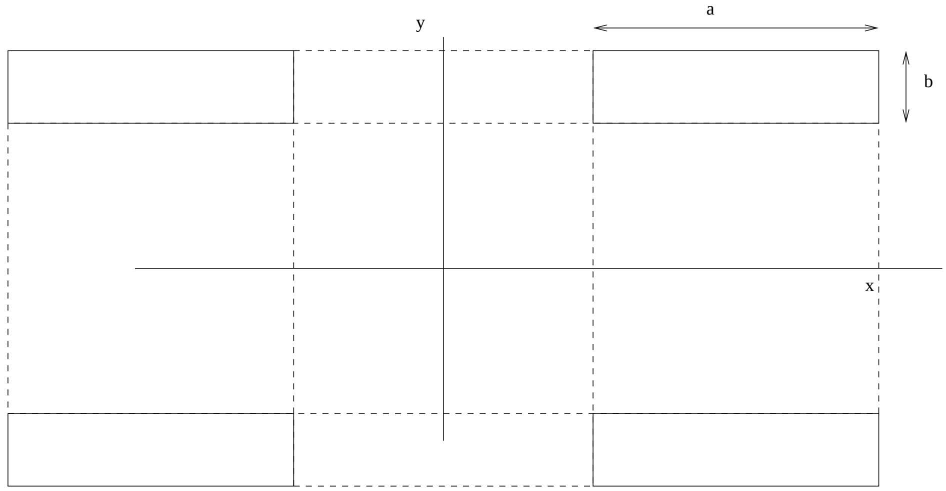

FIG. 3: Decomposition of the off-axis case into four symmetric cases.

Two variants of this geometry exist. Figure 4 shows the case where the signs of the Cartesian coordinates four corners of the plate differ in either the x-value or the y-value but not both. We still adopt the sign convention that

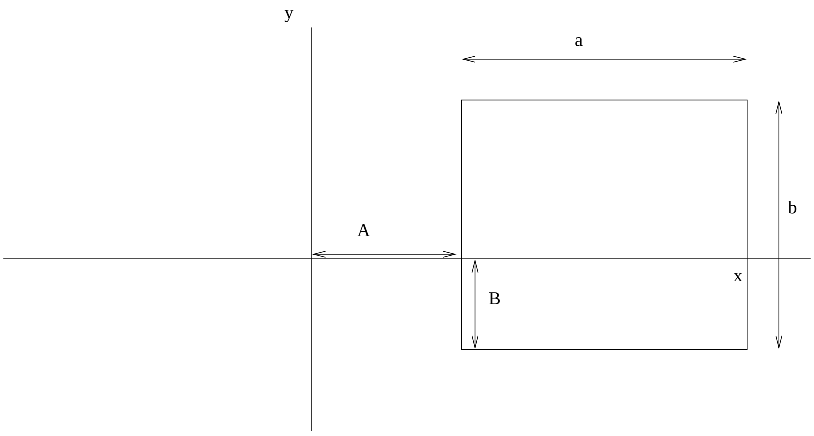

FIG. 4: Off-axis 2-quadrant case, Eq. (A3).

$0 < B \le b/2$ are positive. A similar calculation as above leads to the formula

$$\frac{\Omega(2(A+a),2(b-B),d) - \Omega(2A,2(b-B),d) + \Omega(2(A+a),2B,d) - \Omega(2A,2B,d)}{4} \tag{A3}$$

for the solid angle of the $a \times b$ rectangle in Fig. 4.

In the second variant, the line of sight hits the plate off-center (Fig. 5) with $0 \le A \le a/2$ and $0 \le B \le b/2$. The solid angle covered by the rectangle $a \times b$ becomes

$$\frac{\Omega(2(a-A),2(b-B),d) + \Omega(2A,2(b-B),d) + \Omega(2(a-A),2B,d) + \Omega(2A,2B,d)}{4}. \tag{A4}$$

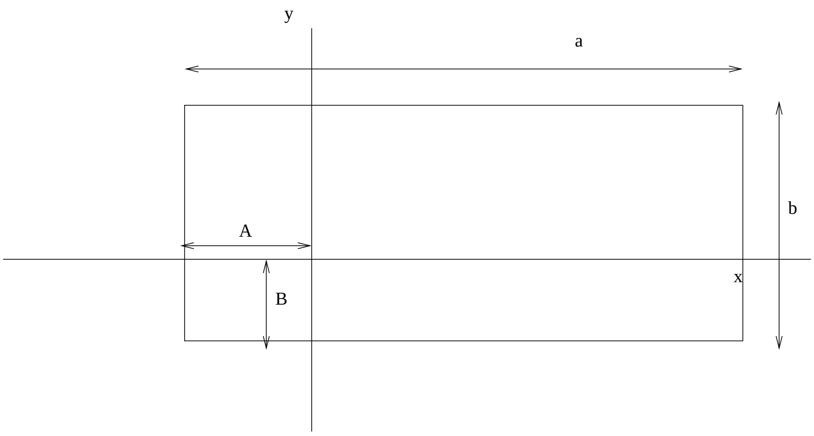

FIG. 5: Off-axis 4-quadrant case, Eq. (A4).

Appendix B: Solid Angle of the Sphere Cap

The equivalent calculation for a plane circle of radius R (area πR^2) at distance d (and of the solid angle of a sphere cap at distance $\sqrt{d^2+R^2}$) is

$$\Omega = 2\pi \int_0^{\arctan\frac{R}{d}} \sin\theta\, d\theta = -2\pi \int_1^{\cos\arctan\frac{R}{d}} dt = 2\pi\left(1 - \frac{1}{\sqrt{1+(\frac{R}{d})^2}}\right) = 2\pi(1-\cos\phi), \tag{B1}$$

where ϕ is half of the cone angle (ie, the angle between the circle center and circle rim seen by the observer). For small ϕ (small R/d), the Taylor expansions are

$$\Omega \approx \pi(\frac{R}{d})^2 - \frac{3\pi}{4}(\frac{R}{d})^4 + \frac{5\pi}{8}(\frac{R}{d})^6 \quad \text{(B2)}$$

$$\approx \pi\phi^2 - \frac{\pi}{12}\phi^4 + \frac{\pi}{360}\phi^6. \quad \text{(B3)}$$

Circles from more general viewing angles are discussed elsewhere [6–9].

Triangulation of more complicated surfaces to finite elements leads to the solid angle of triangles [10–12].

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