Correlations in β decay

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This note is an attempt to generalise the existing results for correlations in cascade transitions. In particular we focus on the case of β-delayed α decay where the intermediate state is best described as a superposition of eigenstates of the nuclear Hamiltonian. Such a generalisation requires knowledge of the transition matrix elements and some understanding of the density operator formalism. The density operator formalism plays an important part in the theoretical derivation of various correlation distributions in nuclear reactions and β decay in particular. In some treatments, however, the formalism is introduced in a somewhat unclear fashion [2], and for this reason I thought it useful to include a section where the basic concepts are introduced. More complete introductions can be found, for instance in [1] or [3].

1. *Description of states*

A quantum system is said to be in a *pure* state if it is possible to write its state vector as $|\alpha\rangle$. Such a state can also be described by the corresponding *density operator*, having the form

$$\rho = |\alpha\rangle\langle\alpha|. \tag{1}$$

Sometimes it is convenient to expand the density operator in terms of some, complete, basis $\{|a_k\rangle\}$:

$$\rho = \sum_{kk'} |a_k\rangle\langle a_k|\rho|a_{k'}\rangle\langle a_{k'}| = \sum_{kk'} |a_k\rangle\langle a_{k'}|\,\rho(k,k'), \tag{2}$$

where $\rho(k,k')$ is the *density matrix* describing the state. One strength of the density operator formalism is that it is possible not only to describe pure quantum states, but also *mixed* states of the form

$$\rho = p(\alpha)|\alpha\rangle\langle\alpha| + p(\beta)|\beta\rangle\langle\beta|. \tag{3}$$

Here, $p(\alpha)$ and $p(\beta)$ denote the probability that the system is in the state $|\alpha\rangle$ or $|\beta\rangle$, respectively. It would not be possible to treat such a state in the ordinary state

vector notation. Clearly, the probabilities must add up to 1, which immediately leads to the useful relation

$$\mathrm{tr}(\rho) = 1, \tag{4}$$

i.e., the diagonal elements of the density matrix must add up to unity.

The density operator formalism is also useful when dealing with compound systems. If we have a compound system consisting of two sub-systems, A and B, which is described by the density operator ρ_{AB}, it is possible to find the state of each sub-system by "tracing out" the other system, i.e.

$$\rho_A = \mathrm{tr}_B(\rho_{AB}), \tag{5}$$

where tr_B denotes the *partial trace* over system B. ρ_A is now the *reduced density operator* for system A. The idea can be illustrated by a pure system of two entangled spins:

$$\begin{aligned} &|\psi_{AB}\rangle = |\uparrow_A\downarrow_B\rangle + |\downarrow_A\uparrow_B\rangle \\ \Rightarrow\ &\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| \\ &\quad = |\uparrow_A\downarrow_B\rangle\langle\uparrow_A\downarrow_B| + |\uparrow_A\downarrow_B\rangle\langle\downarrow_A\uparrow_B| + |\downarrow_A\uparrow_B\rangle\langle\uparrow_A\downarrow_B| + |\downarrow_A\uparrow_B\rangle\langle\downarrow_A\uparrow_B| \end{aligned} \tag{6}$$

The density operator for sub-system A is obtained by taking the partial trace over sub-system B.

$$\rho_A = \mathrm{tr}_B(\rho_{AB}) = |\uparrow_A\rangle\langle\uparrow_A| + |\downarrow_A\rangle\langle\downarrow_A|. \tag{7}$$

It is worth noting that while the combined system is in a pure state, the state of each sub-system is a statistical mixture. The only knowledge we have about such a system is the *probability* with which it occupies each of the possible states.

2. *Transitions and cascades*

Consider a system which undergoes a transition due to the operator T_1, i.e. a final state can be expressed in terms of an initial state like $|f\rangle = T_1|i\rangle$. If the initial state is described by the density operator ρ_A, the final state becomes

$$\rho_B = T_1\rho_A T_1^\dagger = \sum_{kk'} T_1|a_k\rangle\langle a_{k'}|T_1^\dagger\,\rho_A(k,k'). \tag{8}$$

According to [1] this is only strictly true if T_1 is unitary. If T_1 is not unitary, the state ρ_B is not normalised and does not fulfill the trace condition in eq. (4) (however the norm of ρ_B might still contain useful information).

Again, it is straightforward to express the resulting state in the most convenient basis, as long as it forms a complete set $\{|b_l\rangle\}$:

$$\begin{aligned}\rho_B &= \sum_{ll'} |b_l\rangle\langle b_l|\rho_B|b_{l'}\rangle\langle b_{l'}| \\ &= \sum_{ll'}\sum_{kk'} |b_l\rangle\langle b_l|T_1|a_k\rangle\langle a_{k'}|T_1^\dagger|b_{l'}\rangle\langle b_{l'}|\rho_A(k,k') \\ &= \sum_{ll'}\sum_{kk'} |b_l\rangle\langle b_{l'}|\underbrace{\langle b_l|T_1|a_k\rangle\langle b_{l'}|T_1|a_{k'}\rangle^*}_{\rho_1}\rho_A(k,k'). \end{aligned} \tag{9}$$

The brace in the last line indicates ρ_1, which is also defined in eq. (7.7) of [2] and in eq. (47) of [3] (although in slightly different notations) as

$$\rho_1(l,l',k,k') = \langle b_l|T_1|a_k\rangle\langle b_{l'}|T_1|a_{k'}\rangle^* \tag{10}$$

In [2, 3] ρ_1 is introduced as a density matrix, which I find a bit confusing. As far as I can see, it is not a density matrix in the ordinary sense, since it does not describe a quantum state. In the following we speak, however, of ρ_1 as a density matrix, since this appears to be the generally accepted term. We can use ρ_1 to write a conventional density operator for the daughter state. If we define

$$\rho_B(l,l') = \sum_{kk'} \rho_1(l,l',k,k')\rho_A(k,k'), \tag{11}$$

then

$$\rho_B = \sum_{ll'} |b_l\rangle\langle b_{l'}|\rho_B(l,l'). \tag{12}$$

It is possible to use this formalism to describe cascade transitions, where the final state of the first transition acts as intial state of the second transition[1]. Let the operator T_2 be responsible for the second transition. The density operator for the final state is then

$$\rho_C = T_2\rho_B T_2^\dagger = \sum_{ll'} T_2|b_l\rangle\langle b_{l'}|T_2^\dagger\rho_B(l,l'). \tag{13}$$

We now express ρ_C in terms of a complete basis $\{|c_m\rangle\}$. Following the same procedure as in eqs. (2) and (9), we obtain

$$\begin{aligned}\rho_C &= \sum_{mm'} |c_m\rangle\langle c_m|\rho_C|c_{m'}\rangle\langle c_{m'}| \\ &= \sum_{mm'}\sum_{ll'} |c_m\rangle\langle c_m|T_2|b_l\rangle\langle b_{l'}|T_2^\dagger|c_{m'}\rangle\langle c_{m'}|\rho_B(l,l') \\ &= \sum_{mm'}\sum_{ll'} |c_m\rangle\langle c_{m'}|\underbrace{\langle c_m|T_2|b_l\rangle\langle c_{m'}|T_2|b_{l'}\rangle^*}_{\rho_2}\rho_B(l,l'), \end{aligned} \tag{14}$$

[1] Assuming there are no extranuclear fields that modify the intermediate state between the transitions.

where ρ_2 is defined in the same way as ρ_1 of eq. (10):

$$\rho_2(m,m',l,l') = \langle c_m|T_2|b_l\rangle\langle c_{m'}|T_2|b_{l'}\rangle^*. \tag{15}$$

With these definitions we can combine the results in eqs. (11) and (14) and write the density matrix for the final state in terms of the density matrix of the initial state, ρ_A, and the two transition density matrices, ρ_1 and ρ_2:

$$\rho_C(m,m') = \sum_{ll'}\sum_{kk'} \rho_2(m,m',l,l')\rho_1(l,l',k,k')\rho_A(k,k'). \tag{16}$$

It is straightforward to extend the formalism to even more complicated situations involving triple- or quadrupel-cascades. A couple of examples immediately spring to mind:

$$^{12}\mathrm{N} \xrightarrow{\beta} {}^{12}\mathrm{C}^* \xrightarrow{\alpha} {}^{8}\mathrm{Be} \xrightarrow{\alpha} {}^{4}\mathrm{He}$$
$$^{11}\mathrm{B} \xrightarrow{p} {}^{12}\mathrm{C}^{**} \xrightarrow{\gamma} {}^{12}\mathrm{C}^* \xrightarrow{\alpha} {}^{8}\mathrm{Be} \xrightarrow{\alpha} {}^{4}\mathrm{He}.$$

3. *The correlation function*

According to the definition of the density matrix the probability of finding the daughter system in the state $|c_m\rangle$ is given by

$$p(c_m) = \langle c_m|\rho_C|c_m\rangle. \tag{17}$$

If we consider nuclear transitions involving the emission (or absorption) of some kind of radiation, then ρ_C is not only dependent on the quantum numbers of the nuclear system, but also on the properties of the radiation; energy, momentum and polarisation. To be more explicit, we could therefore write for the double-cascade

$$p(c_m) = \langle c_m|\rho_C(\mathbf{k}_1,\mathbf{k}_2)|c_m\rangle, \tag{18}$$

where $\mathbf{k}_1, \mathbf{k}_2$ denote the properties of the radiations involved in the cascade. To calculate the probability of finding the daughter system in *any* state, given $\mathbf{k}_1$ and $\mathbf{k}_2$, we trace over all possible final states:

$$W(\mathbf{k}_1,\mathbf{k}_2) = \sum_m \langle c_m|\rho_C(\mathbf{k}_1,\mathbf{k}_2)|c_m\rangle = \mathrm{tr}(\rho_C). \tag{19}$$

This is the general formula for the correlation function in a double-cascade. If some properties of the radiation are not observed, the expression should be integrated/summed over those unobserved properties.

4. *Nuclear β decay*

In nuclear β decay we are considering transitions from an initial state, which is expressed in terms of angular momentum quantum numbers I_A and m_A, to a final state which contains the daughter nucleus, described by I_B and m_B, as well as the two leptons, described by their momenta, $\mathbf{p}_e$, $\mathbf{p}_\nu$, and their spins m_e and m_ν. This means that the initial and final states are often expressed in terms of basis kets of the types

$$\begin{aligned} &|I_A m_A\rangle && \text{(basis for initial system)} \\ &|I_B m_B \mathbf{p}_e \mathbf{p}_\nu m_e m_\nu\rangle && \text{(basis for final system)} \end{aligned} \tag{20}$$

Typically the initial state is completely mixed, and describes a statistical ensemble of either unoriented or polarised nuclei. The density matrices for the two possibilities are[2]

$$\rho_A(m_A, m'_A) = (2I_A+1)^{-1}\delta_{m_A m'_A} \tag{21}$$

or

$$\rho_A(m_A, m'_A) = p(m_A)\delta_{m_A m'_A}, \tag{22}$$

where the numbers $p(m_A)$ are the population probabilities indicating the polarisation of the ensemble, and we have $\sum_{m_A} p(m_A) = 1$. In [2, 3, 4] many useful calculations are found, however, only transitions to final states of sharp angular momentum are treated. In the following we try to generalise the results to cases where the daughter state of the β decay does not necessarily have a well-defined nuclear spin.

We take the transition matrix element for β^- decay from eq. (6.174) of [2] as our starting point:

$$\begin{aligned} T_{\beta^-} = &\frac{G_\beta}{4\pi}\sum_{KM}\sum_{\substack{\varkappa_e\mu_e \\ \varkappa_\nu\mu_\nu}}(-1)^{I_B-m_B+K+M+j_e-\mu_e+\mu_\nu-l_\nu+\frac{1}{2}} \\ &\times\left[(2I_A+1)(2K+1)\right]^{\frac{1}{2}}\begin{pmatrix} I_B & K & I_A \\ -m_B & M & m_A\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu\end{pmatrix} \\ &\times a^*_{\varkappa_e\mu_e} b^*_{\varkappa_\nu\mu_\nu}\alpha_{\varkappa_e}\left[M_K(k_e,k_\nu)+\operatorname{sign}(\varkappa_e)m_K(k_e,k_\nu)\right]. \end{aligned} \tag{23}$$

Here, the $\varkappa$'s and μ's are quantum numbers for the lepton wave functions, and

$$\varkappa = \begin{cases} l & \text{for } j = l-\frac{1}{2} \\ -(l+1) & \text{for } j = l+\frac{1}{2}\end{cases} \tag{24}$$

[2]The expression in eq. (21) deviate from the expression in [2] by a factor $(2I_A+1)^{-1}$.

and $k = |\varkappa|$. The quantities $a_{\varkappa\mu}$ and $b_{\varkappa\mu}$ are the coefficients of the partial wave expansion of the plane wave lepton states, and they are given explicitly in eqs. (6.115b) and (6.117) of [2] as

$$a_{\varkappa_e\mu_e} = \frac{4\pi}{\sqrt{2}}\frac{1}{p_e}\langle l_e\tfrac{1}{2}(\mu_e - m_e)m_e|j_e\mu_e\rangle Y_{l_e}^{\mu_e - m_e *}(\hat{p}_e)\exp(i\Delta_{\varkappa_e})$$
$$b_{\varkappa_\nu\mu_\nu} = \frac{4\pi}{\sqrt{2}}\langle l_\nu\tfrac{1}{2}(\mu_\nu - m_\nu)m_\nu|j_\nu\mu_\nu\rangle Y_{l_\nu}^{\mu_\nu - m_\nu *}(\hat{p}_\nu) \tag{25}$$

$\alpha_\varkappa$ and $\Delta_\varkappa$ are the Coulomb-amplitude and phase-shift of the electron radial wave function, respectively, see [5] or sec. 4.1 of [2] for a discussion of these quantities. The quantum number, K, is the "multipolarity" of the transition, i.e. the total angular momentum carried away by the leptons[3] and, finally, we have the quantities $M_K(k_e,k_\nu)$ and $m_K(k_e,k_\nu)$, which are defined in eqs. (6.171) and (6.172) of [2]. The expressions for $M_K(k_e,k_\nu)$ and $m_K(k_e,k_\nu)$ are rather formidable, and they are formally written in terms of the so-called *nuclear form-factor coefficients*, which appear from the relativistic treatment of nuclear β decay. If certain approximations are made, however, $M_K(k_e,k_\nu)$ and $m_K(k_e,k_\nu)$ are simply related to the non-relativistic matrix elements of Fermi and Gamow-Teller transitions.

To proceed, we use the definition of the transition density matrix in eq. (10) and combine it with our formula for T_{β^-} (note that in the notation of the earlier sections, $T_{\beta^-} = \langle b_l|T_{\beta^-}|a_k\rangle$, such that $\rho_{\beta^-} = \langle b_l|T_{\beta^-}|a_k\rangle\langle b_{l'}|T_{\beta^-}|a_{k'}\rangle^*$)

$$\begin{aligned}
\rho_{\beta^-} =& \frac{G_\beta^2}{(4\pi)^2}\sum_{KM}\sum_{K'M'}\sum_{\substack{\varkappa_e\mu_e\\ \varkappa_\nu\mu_\nu}}\sum_{\substack{\varkappa'_e\mu'_e\\ \varkappa'_\nu\mu'_\nu}}(-1)^{I_B+I'_B-m_B-m'_B+K+K'+M+M'+j_e+j'_e-\mu_e-\mu'_e+\mu_\nu+\mu'_\nu}\\
&\times(-1)^{-l_\nu-l'_\nu+1}(2I_A+1)\big[(2K+1)(2K'+1)\big]^{\frac{1}{2}}\begin{pmatrix} I_B & K & I_A\\ -m_B & M & m_A\end{pmatrix}\\
&\times\begin{pmatrix} I'_B & K' & I_A\\ -m'_B & M' & m'_A\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu\\ -\mu_e & -M & -\mu_\nu\end{pmatrix}\begin{pmatrix} j'_e & K' & j'_\nu\\ -\mu'_e & -M' & -\mu'_\nu\end{pmatrix}\\
&\times a^*_{\varkappa_e\mu_e}a_{\varkappa'_e\mu'_e}b^*_{\varkappa_\nu\mu_\nu}b_{\varkappa'_\nu\mu'_\nu}\alpha_{\varkappa_e}\alpha_{\varkappa'_e}\big[M_K(k_e,k_\nu)M'_{K'}(k'_e,k'_\nu)\\
&+\mathrm{sign}(\varkappa_e)m_K(k_e,k_\nu)M'_{K'}(k'_e,k'_\nu)+\mathrm{sign}(\varkappa'_e)M_K(k_e,k_\nu)m'_{K'}(k'_e,k'_\nu)\\
&+\mathrm{sign}(\varkappa_e)\mathrm{sign}(\varkappa'_e)m_K(k_e,k_\nu)m'_{K'}(k'_e,k'_\nu)\big].
\end{aligned}\tag{26}$$

This formula looks very similar to eq. (7.1) of [2], but we have allowed for the possibility that $I_B \neq I'_B$. Also, we have put primes on some of the M_K and m_K parameters to allow for several nuclear daughter states to be populated in the transition. The next step is to substitute the $a_{\varkappa\mu}$'s and $b_{\varkappa\mu}$'s with the expressions in eq. (25) and simplify. Following eq. (7.2) to (7.5) in [2] we arrive at a more transparent

[3] It is also the rank of the spherical tensor operator which is responsible for the transition.

form of the density matrix

$$\begin{aligned}
\rho_{\beta^-} =& \frac{G_\beta^2 \pi}{p_e^2} \sum_{KM} \sum_{K'M'} \sum_{\substack{\varkappa_e \mu_e \\ \varkappa_\nu \mu_\nu}} \sum_{\substack{\varkappa'_e \mu'_e \\ \varkappa'_\nu \mu'_\nu}} \sum_{km} \sum_{k'm'} (-1)^{I_B + I'_B - m_B - m'_B + K + K' + M + M' + j_e + j'_e} \\
& \times (-1)^{l_e + l'_e - 1 + m'_e + m'_\nu - \mu'_e - \mu'_\nu} (2I_A + 1) \big[(2K+1)(2K'+1)(2j_e+1)(2j'_e+1) \\
& \times (2j_\nu + 1)(2j'_\nu + 1)(2l_e+1)(2l'_e+1)(2l_\nu+1)(2l'_\nu+1)(2k+1)(2k'+1) \big]^{\frac{1}{2}} \\
& \times \begin{pmatrix} I_B & K & I_A \\ -m_B & M & m_A \end{pmatrix} \begin{pmatrix} I'_B & K' & I_A \\ -m'_B & M' & m'_A \end{pmatrix} \begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu \end{pmatrix} \\
& \times \begin{pmatrix} j'_e & K' & j'_\nu \\ -\mu'_e & M' & -\mu'_\nu \end{pmatrix} \begin{pmatrix} l_e & \frac{1}{2} & j_e \\ \mu_e - m_e & m_e & -\mu_e \end{pmatrix} \begin{pmatrix} l'_e & \frac{1}{2} & j'_e \\ \mu'_e - m'_e & m'_e & -\mu'_e \end{pmatrix} \\
& \times \begin{pmatrix} l_e & l'_e & k \\ \mu_e - m_e & \mu'_e - m'_e & m \end{pmatrix} \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_\nu & \frac{1}{2} & j_\nu \\ \mu_\nu - m_\nu & m_\nu & -\mu_\nu \end{pmatrix} \\
& \times \begin{pmatrix} l'_\nu & \frac{1}{2} & j'_\nu \\ \mu'_\nu - m'_\nu & m'_\nu & -\mu'_\nu \end{pmatrix} \begin{pmatrix} l_\nu & l'_\nu & k' \\ \mu_\nu - m_\nu & m'_\nu - \mu'_\nu & m' \end{pmatrix} \begin{pmatrix} l_\nu & l'_\nu & k' \\ 0 & 0 & 0 \end{pmatrix} \\
& \times Y_k^{m*}(\hat{p}_e) Y_{k'}^{m'*}(\hat{p}_\nu) \alpha_{\varkappa_e} \alpha_{\varkappa'_e} \big[M_K(k_e, k_\nu) M'_{K'}(k'_e, k'_\nu) \\
& + \operatorname{sign}(\varkappa_e) m_K(k_e, k_\nu) M'_{K'}(k'_e, k'_\nu) + \operatorname{sign}(\varkappa'_e) M_K(k_e, k_\nu) m'_{K'}(k'_e, k'_\nu) \\
& + \operatorname{sign}(\varkappa_e) \operatorname{sign}(\varkappa'_e) m_K(k_e, k_\nu) m'_{K'}(k'_e, k'_\nu) \big] \exp\big[-i \big(\Delta_{\varkappa_e} - \Delta_{\varkappa'_e} \big) \big]. \qquad (27)
\end{aligned}$$

This is the complete density matrix for a nuclear β decay. If some properties remain unobserved they should be traced out of the expression, a process which usually provides great simplification.

We now consider the case where neither of the lepton polarisations are observed. In this situation the density matrix is modified

$$\rho_{\beta^-} \longrightarrow \sum_{m_e m'_e} \sum_{m_\nu m'_\nu} \rho_{\beta^-}(m_e, m'_e, m_\nu, m'_\nu) \delta_{m_e m'_e} \delta_{m_\nu m'_\nu}. \qquad (28)$$

These sums, together with the summation over μ_e, μ'_e, μ_ν and μ'_ν, are carried out in eqs. (7.9) to (7.11) in [2]. The procedure is trivial in the sense that it only requires knowledge about the properties of the Wigner $3j$-symbols (see appendix B), and we find

$$\begin{aligned}
\rho_{\beta^-} =& G_\beta^2 \pi F_0 \sum_{KM} \sum_{K'M'} \sum_{Ns} \sum_{km} \sum_{k'm'} (-1)^{I_B + I'_B - m_B - m'_B + N + K + K' + M'} \\
& \times (2I_A + 1)(2N+1)^{\frac{1}{2}} \begin{pmatrix} I_B & K & I_A \\ -m_B & M & m_A \end{pmatrix} \begin{pmatrix} I'_B & K' & I_A \\ -m'_B & M' & m'_A \end{pmatrix} \\
& \times \begin{pmatrix} N & K' & K \\ s & -M' & M \end{pmatrix} \begin{pmatrix} k' & N & k \\ m' & s & m \end{pmatrix} a^N_{kk'}(K, K') Y_k^{m*}(\hat{p}_e) Y_{k'}^{m'*}(\hat{p}_\nu). \qquad (29)
\end{aligned}$$

Here, we have also introduced the factor $a^N_{kk'}(K,K')$ which depends only on the β transition:

$$a^N_{kk'}(K,K') = \frac{1}{F_0 p_e^2} \sum_{\varkappa_e \varkappa'_e} \sum_{\varkappa_\nu \varkappa'_\nu} g^{kk'N}_{KK'}(\varkappa_e, \varkappa'_e, \varkappa_\nu, \varkappa'_\nu) \alpha_{\varkappa_e} \alpha_{\varkappa'_e} \big[M_K(k_e,k_\nu) M'_{K'}(k'_e,k'_\nu) + \mathrm{sign}(\varkappa_e) m_K(k_e,k_\nu) M'_{K'}(k'_e,k'_\nu) + \mathrm{sign}(\varkappa'_e) M_K(k_e,k_\nu) m'_{K'}(k'_e,k'_\nu) + \mathrm{sign}(\varkappa_e)\mathrm{sign}(\varkappa'_e) m_K(k_e,k_\nu) m'_{K'}(k'_e,k'_\nu) \big] \exp\big[-i(\Delta_{\varkappa_e} - \Delta_{\varkappa'_e})\big], \quad (30)$$

where the geometrical factor, $g^{kk'N}_{KK'}(\varkappa_e, \varkappa'_e, \varkappa_\nu, \varkappa'_\nu)$, is defined in eq. (7.14) of [2]. Again, the primes on some of the M_K's and m_K's make our definition of $a^N_{kk'}(K,K')$ slightly more general than the standard definition. The remaining factors in eq. (29) depend only on the quantum numbers of the nuclear states and the Fermi function, F_0.

5. *Emission of α particles*

In this section we attempt to apply the density matrix formalism to another kind of transitions, namely α transitions. The idea is to find an expression equivalent to eq. (29) and eventually combine the results in a correlation function for β-delayed α decays. We start from the general transition matrix element in eq. (59) of [3], describing a transition from an initial state with spin, I_B, to a final state of spin I_C:

$$T_\alpha = \sum_{LM\mu} (-1)^{-I_c+L-m_B} \begin{pmatrix} I_C & L & I_B \\ m_C & M & -m_B \end{pmatrix} \langle 0\sigma | L\mu\pi\rangle \langle I_C || L\pi || I_B\rangle \mathscr{D}^{L*}_{M\mu}(\mathbf{z}\to\mathbf{p}). \quad (31)$$

In the above formula, $\langle 0\sigma|L\mu\pi\rangle$ is the eigenfunction of radiation emitted along the z-axis corresponding to definite values of angular momentum, its projection and parity.[4] Furthermore we have the reduced matrix element, $\langle I_C||L\pi||I_B\rangle$, which can be chosen to be real if the transition operator is symmetric under time reversal [3]. Finally, $\mathscr{D}^{*L}_{M\mu}$ is an element of the rotation matrix, see for instance appendix A.

It is straightforward to construct a transition density matrix from eq. (31).

$$\rho_\alpha = \sum_{LM\mu} \sum_{L'M'\mu'} (-1)^{-2I_C+L+L'-m_B-m'_B} \langle 0\sigma|L\mu\pi\rangle \langle 0\sigma'|L'\mu'\pi'\rangle^* \times \langle I_C||L\pi||I_B\rangle \langle I_C||L'\pi||I'_B\rangle^* \begin{pmatrix} I_C & L & I_B \\ m_c & M & -m_B \end{pmatrix} \begin{pmatrix} I_C & L' & I'_B \\ m'_C & M' & -m'_B \end{pmatrix} \times \mathscr{D}^{L*}_{M\mu}(\mathbf{z}\to\mathbf{p}) \mathscr{D}^{L'*}_{M'\mu'}(\mathbf{z}\to\mathbf{p}) \quad (32)$$

[4] This is probably not very well stated. See sec. 3.2.1 and 3.5 of [3] and figure out the correct interpretation.

The elements of the rotation matrix have many useful properties, some of which are summarised appendix A. Specifically we can write the product of two matrix elements in terms of single matrix elements (eq. (74)), and we find

$$\mathscr{D}^{L*}_{M\mu}(\mathbf{z}\to\mathbf{p})\mathscr{D}^{L'*}_{M'\mu'}(\mathbf{z}\to\mathbf{p}) = (-1)^{M-\mu}\sum_k (2k+1)(-1)^{\tau+N}\begin{pmatrix} L & L' & k \\ M & -M' & N\end{pmatrix}\begin{pmatrix} L & L' & k \\ \mu & -\mu' & \tau\end{pmatrix}\mathscr{D}^{k}_{N\tau}(\mathbf{z}\to\mathbf{p}). \tag{33}$$

Substituting this identity into (32) we get

$$\begin{aligned}\rho_\alpha = \sum_{LM\mu}\sum_{L'M'\mu'}\sum_{kN\tau}(-1)^{-2I_C+L+L'-m_B-m'_B+M-\mu+\tau+N}&\langle 0\sigma|L\mu\pi\rangle\langle 0\sigma'|L'\mu'\pi'\rangle^* \\ &\times\langle I_C||L\pi||I_B\rangle\langle I_C||L'\pi||I'_B\rangle^*(2k+1)\begin{pmatrix} I_C & L & I_B \\ m_C & M & -m_B\end{pmatrix} \\ &\times\begin{pmatrix} I_C & L' & I'_B \\ m'_C & M' & -m'_B\end{pmatrix}\begin{pmatrix} L & L' & k \\ M & -M' & N\end{pmatrix}\begin{pmatrix} L & L' & k \\ \mu & -\mu' & \tau\end{pmatrix}\mathscr{D}^{k}_{N\tau}(\mathbf{z}\to\mathbf{p}),\end{aligned} \tag{34}$$

which is the formula for an α transition from a state without a well-defined angular momentum, denoted I_B, I'_B, to a single state of angular momentum I_C. The sign-factor can be simplified a bit by realising that the third of the $3j$-symbols vanishes unless

$$N = M' - M \quad\Rightarrow\quad M + N = M'. \tag{35}$$

The last $3j$-symbol vanishes unless $\tau = \mu' - \mu$. This means that τ has to be an integer, and we find

$$(-1)^{\tau-\mu} = (-1)^{-\tau-\mu} = (-1)^{-\mu'}. \tag{36}$$

The sign-factor can now be re-written in a slightly more compact way:

$$(-1)^{-2I_C+L+L'-m_B-m'_B+M-\mu+\tau+N} = (-1)^{-2I_C+L+L'-m_B-m'_B+M'-\mu'}. \tag{37}$$

6. *Correlations in β-delayed α decay*

Now that we have found expressions for the transition density matrices of both β and α transitions, we are ready to combine the results and write down the correlation function for a β-α cascade. The initial and final states are specified by their angular momentum quantum numbers, $\{I_A, m_A\}$ and $\{I_C, m_C\}$, respectively, and we assume that the transition can proceed through several intermediate nuclear states, labelled λ_B, with angular momentum quantum numbers $\{I_B, m_B\}$. Note that the states λ_B are assumed to be eigenstates of the nuclear hamiltonian, and they are therefore states of definite angular momentum. This means that λ_B implicitly

specifies I_B, but for the sake of clarity we retain I_B in the formulas below. The standard treaments only consider the possibility of a single intermediate state, and in these treatments only the specification of the magnetic sub-states, m_B, is required [2, 3, 4]. Following eqs. (16) and (19) we write

$$\begin{aligned} W = \mathrm{tr}(\rho_C) = & \sum_{m_C} \sum_{\lambda_B \lambda_B'} \sum_{m_B m_B'} \sum_{m_A m_A'} \rho_\alpha(m_C, m_C, I_B, I_B', m_B, m_B') \\ & \times \rho_{\beta^-}(I_B, I_B', m_B, m_B', m_A, m_A') \rho_A(m_A, m_A') \\ = & \sum_{\lambda_B \lambda_B'} \sum_{m_B m_B'} \left[\sum_{m_C} \rho_\alpha(m_C, m_C, I_B, I_B', m_B, m_B') \right] \\ & \times \left[\sum_{m_A m_A'} \rho_{\beta^-}(I_B, I_B', m_B, m_B', m_A, m_A') \rho_A(m_A, m_A') \right]. \end{aligned} \tag{38}$$

Here, we follow more or less the notation of [2], where the dependence of each density matrix on the various quantum numbers has been made explicit. Keep in mind, however, that ρ_α and ρ_{β^-} also depend on the energies and momenta of the emitted radiations.

To proceed, we evaluate each of the brackets in eq. (38) separately. We start with the second bracket, where we immediately replace ρ_A with the density matrix for an unpolarised ensemble, which we get from eq. (21). Taking the transition density matrix from eq. (29), we need to calculate

$$\begin{aligned} & \sum_{m_A m_A'} \rho_{\beta^-}(I_B, I_B', m_B, m_B', m_A, m_A') \delta_{m_A m_A'} (2I_A + 1)^{-1} = \mathrm{tr}_A(\rho_{\beta^-} \rho_A) \\ & = G_\beta^2 \pi F_0 \sum_{m_A} \sum_{K_1 M_1} \sum_{K_1' M_1'} \sum_{N_1 s_1} \sum_{k_1 m_1} \sum_{k_1' m_1'} (-1)^{I_B + I_B' - m_B - m_B' + N_1 + K_1 + K_1' + M_1'} (2N_1 + 1)^{\frac{1}{2}} \\ & \times \begin{pmatrix} I_b & K_1 & I_A \\ -m_B & M_1 & m_A \end{pmatrix} \begin{pmatrix} I_B' & K_1' & I_A \\ -m_B' & M_1' & m_A \end{pmatrix} \begin{pmatrix} N_1 & K_1' & K_1 \\ s_1 & -M_1' & M_1 \end{pmatrix} \\ & \times \begin{pmatrix} k_1' & N_1 & k_1 \\ m_1' & s_1 & m_1 \end{pmatrix} a^{N_1}_{k_1 k_1'}(K_1, K_1') Y^{m_1 *}_{k_1}(\hat{p}_e) Y^{m_1' *}_{k_1'}(\hat{p}_\nu), \end{aligned} \tag{39}$$

where all quantum numbers and summation indices related to the β transition has been labelled with a "1". In the following we need to do some gymnastics with the Wigner $3j$-symbols, and therefore it is probably a good idea to have a look in appendix B, where some important properties of the Wigner-symbols are listed. We can use eq. (84) to show that

$$\begin{aligned} & \sum_{M_1 M_1' m_A} (-1)^{M_1' - m_B - m_B'} \begin{pmatrix} I_B & K_1 & I_A \\ -m_B & M_1 & m_A \end{pmatrix} \begin{pmatrix} I_B' & K_1' & I_A \\ -m_B' & M_1' & m_A \end{pmatrix} \begin{pmatrix} N_1 & K_1' & K_1 \\ s_1 & -M_1' & M_1 \end{pmatrix} \\ & = (-1)^{I_A + I_B + I_B' + m_B'} \begin{pmatrix} I_B & I_B' & N_1 \\ m_B & -m_B' & s_1 \end{pmatrix} \begin{Bmatrix} I_B & I_B' & N_1 \\ K_1' & K_1 & I_A \end{Bmatrix}. \end{aligned} \tag{40}$$

In obtaining the above result it is necessary to realise that $M_1' - m_A - m_B'$ is an integer, such that $(-1)^{M_1'-m_A-m_B'} = (-1)^{-M_1'+m_A+m_B'}$. Doing the summations over M_1, M_1' and m_A in eq. (39) we find

$$\begin{aligned}\mathrm{tr}_A(\rho_{\beta^-}\rho_A) = & G_\beta^2 \pi F_0 \sum_{K_1K_1'}\sum_{N_1s_1}\sum_{k_1m_1}\sum_{k_1'm_1'}(-1)^{I_A+2I_B+2I_B'+m_B'+N_1+K_1+K_1'}(2N_1+1)^{\frac{1}{2}} \\ & \times \begin{pmatrix} I_B & I_B' & N_1 \\ m_B & -m_B' & s_1 \end{pmatrix} \begin{Bmatrix} I_B & I_B' & N_1 \\ K_1' & K_1 & I_A' \end{Bmatrix} \begin{pmatrix} k_1' & N_1 & k_1 \\ m_1' & s_1 & m_1 \end{pmatrix} \\ & \times a^N_{kk'}(K_1,K_1') Y^{m_1*}_{k_1}(\hat{p}_e) Y^{m_1'*}_{k_1'}(\hat{p}_\nu). \end{aligned} \tag{41}$$

Since I_B and I_B' can only differ by an integer, the sum $I_B + I_B'$ must be an integer. This means that the term $2I_B + 2I_B'$ must be an even number, and we are allowed to ignore it when it appears in the sign factor.

We now focus on the special (although quite common) case in which the neutrino is not observed. This means that we must integrate over $\hat{p}_\nu$. We have

$$\begin{aligned} & \int Y^{m_1'*}_{k_1'}(\hat{p}_\nu)\, d\hat{p}_\nu = \sqrt{4\pi}\, \delta_{k_1'0}\delta_{m_1'0} \\ \Rightarrow \qquad & k_1' = 0 \quad \text{and} \quad m_1' = 0. \end{aligned} \tag{42}$$

These constraints on k_1' and m_1' in turn lead to constraints on N_1 and s_1 through the $3j$-symbol and the identity in eq. (78).

$$\begin{aligned} & \begin{pmatrix} 0 & N_1 & k_1 \\ 0 & s_1 & m_1 \end{pmatrix} = (-1)^{-N_1-m_1}(2k_1+1)^{\frac{1}{2}}\delta_{N_1k_1}\delta_{s_1(-m_1)} \\ \Rightarrow \qquad & N_1 = k_1 \quad \text{and} \quad s_1 = -m_1. \end{aligned} \tag{43}$$

Under these conditions eq. (41) simplifies considerably:

$$\begin{aligned}\mathrm{tr}_A(\rho_{\beta^-}\rho_A) = & 2\pi^{\frac{3}{2}} G_\beta^2 F_0 \sum_{K_1K_1'}\sum_{k_1m_1}(-1)^{I_A+m_B'+K_1+K_1'-m_1} \\ & \times \begin{pmatrix} I_B & I_B' & k_1 \\ m_B & -m_B' & -m_1 \end{pmatrix} \begin{Bmatrix} I_B & I_B' & k_1 \\ K_1' & K_1 & I_A \end{Bmatrix} a^{k_1}_{k_10}(K_1,K_1') Y^{m_1*}_{k_1}(\hat{p}_e). \end{aligned} \tag{44}$$

It is now customary to define the *particle parameters* of the nuclear β transition as

$$b^{(k)}_{KK'} = \frac{a^k_{k0}(K,K') + a^k_{k0}(K',K)}{4(1+\delta_{KK'})} \tag{45}$$

and replace the sum over K and K' with a sum over $K \leq K'$. This definition of the particle parameters is identical to the definition in [2, 4, 5], but differs by some factors from the definition in [3]. Tables 7.2 of [2], 10.8 of [4] and 7 of [5] all

provide explicit expressions for $b^{(k)}_{KK'}$ in terms of M_K's and m_K's, however, we must keep in mind that since our definition of the $a^N_{kk'}(K,K')$-parameters in eq. (30) is somewhat more general than the standard definition, these tables cannot be used directly. We shall see later how to deal with this problem.

In order to bring the β- and the α-part of the density matrix on the same form we replace the spherical harmonic with a matrix element of the rotation operator using eq. (75). After a little rearrangement we find

$$\begin{aligned}\mathrm{tr}_A(\rho_{\beta\text{-}}\rho_A) =& 4\pi G_\beta^2 F_0 \sum_{K_1\le K_1'}\sum_{k_1 m_1}(-1)^{I_A+m_B'+K_1+K_1'-m_1}(2k_1+1)^{\frac{1}{2}} \\ &\times \begin{pmatrix} I_B & I_B' & k_1 \\ m_B & -m_B' & -m_1\end{pmatrix}\begin{Bmatrix} I_B & I_B' & k_1 \\ K_1' & K_1 & I_A\end{Bmatrix} b^{(k_1)}_{K_1K_1'}\mathscr{D}^{k_1}_{m_1 0}(\mathbf{z}\to\mathbf{p}_e). \quad (46)\end{aligned}$$

In the special case where only a single intermediate state is populated in the transition, i.e. when $I_B = I_B'$, the above result should be identical to eq. (7.65) in [2]. This is indeed the case, except for an apparent difference in sign. In [2] the term $-I_A - m_B'$ appears in the sign factor, whereas here we have $I_A + m_B'$. The fact that $I_A + m_B'$ is an integer resolves this apparent discrepancy.

Returning now to eq. (38) we now take a closer look at the expression in the first of the two brackets. Using eq. (34) to replace ρ_α we obtain

$$\begin{aligned}&\sum_{m_C}\rho_\alpha(m_C,m_C,I_B,I_B',m_B,m_B') = \mathrm{tr}_C(\rho_\alpha) \\ &\quad=\sum_{m_C}\sum_{L_2M_2\mu_2}\sum_{L_2'M_2'\mu_2'}\sum_{k_2N_2\tau_2}(-1)^{-2I_C+L_2+L_2'-m_B-m_B'+M_2'-\mu_2'}(2k_2+1) \\ &\quad\times\langle 0\sigma|L_2\mu_2\pi_2\rangle\langle 0\sigma'|L_2'\mu_2'\pi_2'\rangle^*\langle I_C||L_2\pi_2||I_B\rangle\langle I_C||L_2'\pi_2'||I_B'\rangle^* \\ &\quad\times\begin{pmatrix} I_C & L_2 & I_B \\ m_C & M_2 & -m_B\end{pmatrix}\begin{pmatrix} I_C & L_2' & I_B' \\ m_C & M_2' & -m_B'\end{pmatrix}\begin{pmatrix} L_2 & L_2' & k_2 \\ M_2 & -M_2' & N_2\end{pmatrix} \\ &\quad\times\begin{pmatrix} L_2 & L_2' & k_2 \\ \mu_2 & -\mu_2' & \tau_2\end{pmatrix}\mathscr{D}^{k_2}_{N_2\tau_2}(\mathbf{z}\to\mathbf{p}_\alpha), \quad (47)\end{aligned}$$

where we have put the subscript "2" on those quantum numbers, which are related to the second transition in the β-α cascade. Employing eq. (84) we can carry out the summation over m_C, M_2 and M_2':

$$\begin{aligned}&\sum_{m_C M_2 M_2'}(-1)^{I_C-m_B-m_B'+M_2'}\begin{pmatrix} I_C & L_2 & I_B \\ m_C & M_2 & -m_B\end{pmatrix}\begin{pmatrix} I_C & L_2' & I_B' \\ m_C & M_2' & -m_B'\end{pmatrix} \\ &\quad\times\begin{pmatrix} L_2 & L_2' & k_2 \\ M_2 & -M_2' & N_2\end{pmatrix} = (-1)^{m_B'+k_2}\begin{pmatrix} I_B & I_B' & k_2 \\ m_B & -m_B' & N_2\end{pmatrix}\begin{Bmatrix} I_B & I_B' & k_2 \\ L_2' & L_2 & I_C\end{Bmatrix}. \quad (48)\end{aligned}$$

Further simplification can be achieved by replacing the slightly mysterious quantities $\langle 0\sigma|L\mu\tau\rangle$. In sec. 6.1 of [3] it is found that for a spinless particle travelling

along the z-axis the quantities are given by

$$\langle 0\sigma|L\mu\pi\rangle \to \langle 00|L\mu\pi\rangle = \begin{cases} (2L+1)^{\frac{1}{2}}\delta_{\mu 0} & \text{for } Z=0 \\ (2L+1)^{\frac{1}{2}}\exp(-i\sigma_L)\delta_{\mu 0} & \text{for } Z\neq 0 \end{cases} \tag{49}$$

where σ_L is the normal Coulomb phase shift.[5] As a consequence we have $\mu_2 = 0$, $\mu_2' = 0$ and, since $\tau_2 = \mu_2' - \mu_2$, we also have $\tau_2 = 0$. Taking this knowledge into account we can rewrite eq. (47) as

$$\begin{aligned} \mathrm{tr}_C(\rho_\alpha) = \sum_{L_2 L_2'} \sum_{k_2 N_2} & (-1)^{I_C+L_2+L_2'+m_B'+k_2}(2k_2+1)\left[(2L_2+1)(2L_2'+1)\right]^{\frac{1}{2}} \\ & \times \langle I_C||L_2\pi_2||I_B\rangle\langle I_C||L_2'\pi_2'||I_B'\rangle^* \begin{pmatrix} I_B & I_B' & k_2 \\ m_B & -m_B' & N_2 \end{pmatrix} \\ & \times \begin{Bmatrix} I_B & I_B' & k_2 \\ L_2' & L_2 & I_C \end{Bmatrix} \begin{pmatrix} L_2 & L_2' & k_2 \\ 0 & 0 & 0 \end{pmatrix} \exp\left[-i(\sigma_{L_2}-\sigma_{L_2'})\right] \mathscr{D}^{k_2}_{N_2 0}(\mathbf{z}\to\mathbf{p}_\alpha). \end{aligned} \tag{50}$$

Traditionally it is now argued that since the intermediate and final states have well-defined parities, and since the parity of the α radiation is $\pi = (-1)^L$, the values of L and L' can only differ by an even number, i.e. $L = L'$, $L = L'+2$, ... Therefore, $L + L'$ is an even number, and, since the $3j$-symbol vanishes unless $L + L' + k$ is an even number (a consequence of eq. (81)), k must also be an even number. I think, however, that in the present case where we are considering the possibility of populating several intermediate states, possibly of differing parity, that such an argument cannot be made.

We are now in a position to combine our results and write the correlation function for a β-delayed α decay explicitly. Combining eqs. (38), (46) and (50) we obtain

$$\begin{aligned} W = \sum_{\lambda_B \lambda_B'} \sum_{m_B m_B'} \sum_{L_2 L_2'} \sum_{k_2 N_2} \sum_{K_1 \le K_1'} \sum_{k_1 m_1} & (-1)^{-I_A+I_C+L_2+L_2'+k_2+K_1+K_1'-m_1} \\ & \times 4\pi G_\beta^2 F_0 (2k_2+1)\left[(2k_1+2)(2L_2+1)(2L_2'+1)\right]^{\frac{1}{2}} \begin{pmatrix} I_B & I_B' & k_1 \\ m_B & -m_B' & -m_1 \end{pmatrix} \\ & \times \begin{pmatrix} I_B & I_B' & k_2 \\ m_B & -m_B' & N_2 \end{pmatrix} \begin{pmatrix} L_2 & L_2' & k_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} I_B & I_B' & k_1 \\ K_1' & K_1 & I_A \end{Bmatrix} \begin{Bmatrix} I_B & I_B' & k_2 \\ L_2' & L_2 & I_C \end{Bmatrix} \\ & \times \langle I_C||L_2\pi_2||I_B\rangle\langle I_C||L_2'\pi_2'||I_B'\rangle^* b^{(k_1)}_{K_1 K_1'} \exp\left[-i(\sigma_{L_2}-\sigma_{L_2'})\right] \\ & \times \mathscr{D}^{k_1}_{m_1 0}(\mathbf{z}\to\mathbf{p}_e)\mathscr{D}^{k_2}_{N_2 0}(\mathbf{z}\to\mathbf{p}_\alpha). \end{aligned} \tag{51}$$

[5] Probably the inclusion of a resonant phase factor could also be appropriate here. This would be a way of connecting to the R-matrix formalism...?

From eq. (83) it follows that

$$\sum_{m_B m_B'} (2k_2+1) \begin{pmatrix} I_B & I_B' & k_1 \\ m_B & -m_B' & -m_1 \end{pmatrix} \begin{pmatrix} I_B & I_B' & k_2 \\ m_B & -m_B' & N_2 \end{pmatrix} = \delta_{k_1 k_2} \delta_{(-m_1) N_2}. \quad (52)$$

Also, using eqs. (71) to (73), we can show that

$$\sum_m (-1)^m \mathscr{D}^k_{m0}(\mathbf{z} \to \mathbf{p}_\beta) \mathscr{D}^k_{(-m)0}(\mathbf{z} \to \mathbf{p}_\alpha) = \mathscr{D}^k_{00}(\mathbf{p}_\alpha \to \mathbf{p}_e) = P_k(\cos\theta_{\beta\alpha}). \quad (53)$$

With these results we get our final expression for the correlation function

$$\begin{aligned} W = {} & 4\pi G_\beta^2 F_0 \sum_k \sum_{\lambda_B \lambda_B'} \sum_{L_2 L_2'} \sum_{K_1 \le K_1'} (-1)^{-I_A+I_C+L_2+L_2'+K_1+K_1'+k} \\ & \times \left[(2k+2)(2L_2+1)(2L_2'+1)\right]^{\frac{1}{2}} \begin{pmatrix} L_2 & L_2' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} I_B & I_B' & k \\ K_1' & K_1 & I_A \end{Bmatrix} \begin{Bmatrix} I_B & I_B' & k \\ L_2' & L_2 & I_C \end{Bmatrix} \\ & \times \langle I_C || L_2 \pi_2 || I_B \rangle \langle I_C || L_2' \pi_2' || I_B' \rangle^* b^{(k)}_{K_1 K_1'} \exp\left[-i(\sigma_{L_2} - \sigma_{L_2'})\right] P_k(\cos\theta_{\beta\alpha}). \end{aligned} \quad (54)$$

It may not be immediately clear how to extract any conclusions from this correlation function, which still appear somewhat complicated. It is possible, however, to write down a list of selection rules imposed by the Wigner-symbols which will come in handy, when we later explore the correlation function in specific examples.

$$\begin{aligned} I_B' &= |I_B - k|, \ldots, I_B + k \\ K_1' &= |K_1 - k|, \ldots, K_1 + k \\ K_1 &= |I_A - I_B|, \ldots, I_A + I_B \\ K_1' &= |I_A - I_B'|, \ldots, I_A + I_B' \\ L_2' &= |L_2 - k|, \ldots, L_2 + k \\ L_2 &= |I_C - I_B|, \ldots, I_C + I_B \\ L_2' &= |I_C - I_B'|, \ldots, I_C + I_B' \\ L_2 + L_2' + k &= \text{an even number.} \end{aligned} \quad (55)$$

7. *Particle parameters*

As mentioned earlier the standard formulas for the particle parameters, $b^{(k)}_{KK'}$, cannot be used directly in our case. In this section we take a closer look at the definition of the parameters and derive the expressions for a few special cases.

From eq. (30) it is possible to derive a formula for the special combination of indices $N = k$ and $k' = 0$:

$$\begin{aligned} a^k_{k0}(K,K') = \frac{2}{p_e^2 F_0} \sum_{x_e x'_e} \sum_{k_\nu} f^k_{KK'}(x_e, x'_e, k_\nu) \alpha_{x_e} \alpha_{x'_e} \big[& M_K(k_e, k_\nu) M'_{K'}(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e) m_K(k_e, k_\nu) M'_{K'}(k'_e, k_\nu) + \mathrm{sign}(x'_e) M_K(k_e, k_\nu) m'_{K'}(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e)\mathrm{sign}(x'_e) m_K(k_e, k_\nu) m'_{K'}(k'_e, k_\nu) \big] \exp\left[-i\left(\Delta_{x_e} - \Delta_{x'_e}\right)\right], \end{aligned} \tag{56}$$

where the coefficients are given by eq. (7.27) of [2] or eq. (10.30) of [4] as

$$\begin{aligned} f^{(k)}_{KK'}(x_e, x'_e, k_\nu) = (-1)^{K+K'+j_e+j'_e+j_\nu+\frac{1}{2}} & \big[(2K+1)(2K'+1)(2k+1)(2j_e+1)(2j'_e+1) \\ & \times (2l_e+1)(2l'_e+1)\big]^{\frac{1}{2}} \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_e & l'_e & k \\ j'_e & j_e & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} K & K' & k \\ j'_e & j_e & j_\nu \end{Bmatrix}. \end{aligned} \tag{57}$$

These coefficients are tabulated in Table 7.1 of [2]. If we take the definition of the particle parameters in eq. (45) it is possible to use eq. (56) to write the parameters in terms of the M_K and m_K quantities:

$$\begin{aligned} b^{(k)}_{KK'} = \frac{1}{2p_e^2 F_0} \frac{1}{1+\delta_{KK'}} \sum_{x_e x'_e} \sum_{k_\nu} \Big[& f^{(k)}_{KK'}(x_e, x'_e, k_\nu) \alpha_{x_e} \alpha_{x'_e} \Big\{ M_K(k_e, k_\nu) M'_{K'}(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e) m_K(k_e, k_\nu) M'_{K'}(k'_e, k_\nu) + \mathrm{sign}(x'_e) M_K(k_e, k_\nu) m'_{K'}(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e)\mathrm{sign}(x'_e) m_K(k_e, k_\nu) m'_{K'}(k'_e, k_\nu) \Big\} \exp\left[-i\left(\Delta_{x_e} - \Delta_{x'_e}\right)\right] \\ & + f^{(k)}_{K'K}(x_e, x'_e, k_\nu) \alpha_{x_e} \alpha_{x'_e} \Big\{ M_{K'}(k_e, k_\nu) M'_K(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e) m_{K'}(k_e, k_\nu) M'_K(k'_e, k_\nu) + \mathrm{sign}(x'_e) M_{K'}(k_e, k_\nu) m'_K(k'_e, k_\nu) \\ & + \mathrm{sign}(x_e)\mathrm{sign}(x'_e) m_{K'}(k_e, k_\nu) m'_K(k'_e, k_\nu) \Big\} \exp\left[-i\left(\Delta_{x_e} - \Delta_{x'_e}\right)\right] \Big]. \end{aligned} \tag{58}$$

At this point it should be reiterated that an un-primed quantity is associated with a transition to λ_B, while the primed quantities are associated with transitions to λ'_B.

When dealing with eq. (58) we need to carry out sums over x_e, x'_e and k_ν. Therefore, a question which quickly arises is which values of the summation indices should be included in the sums. To determine this we derive a few selection rules from the Wigner-symbols in eq. (57).

$$\begin{aligned} l'_e &= |l_e - k|, \ldots, l_e + k \\ j'_e &= |j_e - k|, \ldots, j_e + k \\ K' &= |K - k|, \ldots, K + k \\ j_\nu = |K - j_e|, \ldots, K + j_e \quad &\text{and} \quad j_\nu = |K' - j'_e|, \ldots, K' + j'_e. \end{aligned} \tag{59}$$

In order to get a better feeling for the procedure, let us take a look at the simple (but not unimportant) case where $k = K = K' = 0$. The allowed combinations of quantum numbers are listed in Table 0.1. In principle there are infinitely many terms in the summation, however, the order of magnitude of the terms vary a lot, and in practice it is only necessary to include a few dominant terms.

Table 0.1: Allowed combinations of the lepton quantum numbers that could be included in the calculation of the particle parameters in eq. (58). Only combinations for the case $k = K = K' = 0$ are listed here! We note that several values of l_ν and $\varkappa_\nu$ result in the same k_ν.

l_e	j_e	$\varkappa_e$	k_e	l'_e	j'_e	$\varkappa'_e$	k'_e	l_ν	j_ν	$\varkappa_\nu$	k_ν
0	$\frac{1}{2}$	-1	1	0	$\frac{1}{2}$	-1	1	0/1	$\frac{1}{2}$	-1/1	1
1	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	1	0/1	$\frac{1}{2}$	-1/1	1
1	$\frac{3}{2}$	-2	2	1	$\frac{3}{2}$	-2	2	1/2	$\frac{3}{2}$	-2/2	2
2	$\frac{3}{2}$	2	2	2	$\frac{3}{2}$	2	2	1/2	$\frac{3}{2}$	-2/2	2
2	$\frac{5}{2}$	-3	3	2	$\frac{5}{2}$	-3	3	2/3	$\frac{5}{2}$	-3/3	3
		⋮				⋮					⋮

According to eqs. (7.42) to (7.47) of [2] each term in eq. (58) is proportional to

$$(p_e R)^{k_e + k'_e - 2} (p_\nu R)^{2k_\nu - 2}, \tag{60}$$

and with

$$0 \leq p_e R \lesssim 0.2 \quad \text{and} \quad 0 \leq p_\nu R \lesssim 0.2, \tag{61}$$

the magnitude is roughly determined by the exponent $k_e + k'_e + 2k_\nu - 4$, where the combination with the smallest value of this number will give the dominant term. This means that only the terms in the first two lines of Table 0.1 contribute significantly to the particle parameter $b_{00}^{(0)}$ (allowed Fermi transitions). It is now straightforward to derive direct expressions for the particle parameters. The two simplest examples are

$$\begin{aligned} b_{00}^{(0)} =& L_0 \Big[\{ M_0(1,1) M'_0(1,1) + m_0(1,1) m'_0(1,1) \} \\ & - \mu_1 \gamma_1 \frac{1}{W_e} \{ m_0(1,1) M'_0(1,1) + M_0(1,1) m'_0(1,1) \} \Big] \end{aligned} \tag{62}$$

$$\begin{aligned} b_{11}^{(0)} =& -\sqrt{3} L_0 \Big[M_1(1,1) M'_1(1,1) + m_1(1,1) m'_1(1,1) + M_1(1,2) M'_1(1,2) \\ & - \mu_1 \gamma_1 \frac{1}{W_e} \{ m_1(1,1) M'_1(1,1) + M_1(1,1) m'_1(1,1) + m_1(1,2) M'_1(1,2) \\ & + M_1(1,2) m'_1(1,2) \} + \lambda_2 \big[M_1(2,1) M'_1(2,1) \\ & - \mu_2 \gamma_2 \frac{1}{2 W_e} \{ m_1(2,1) M'_1(2,1) + M_1(2,1) m'_1(2,1) \} \big] \Big], \end{aligned} \tag{63}$$

where we have included the so-called *allowed* and *first-forbidden* contributions. The functions L_0, μ, γ and λ are combinations of the Coulomb amplitudes and phases, and they are defined in eq. (7.51) of [2] in such a way that their value is very close to unity, except for very high Z or very low electron momenta. For numerical values, see Table 2 of [5]. Comparing to the standard particle parameters listed in [2, 4, 5] we see that the expressions are very similar, and that our result reduces to the standard result when $\lambda_B = \lambda'_B$, as indeed it should. In general, terms with the smallest values of K, K' and k are expected to contribute most a transition.

8. *Allowed decays*

A common type of transitions are the *allowed* transitions. These transitions are the ones that satisfy

$$\Delta I = 0, \pm 1 \quad \text{and} \quad \pi_i \pi_f = +1, \tag{64}$$

and the dominant terms are related to only two quantities:

$$\begin{aligned} M_0(1,1) &= {}^V F^{(0)}_{000} = C_V \int \mathbf{1} = C_V M_F \\ M_1(1,1) &= -{}^A F^{(0)}_{101} = -C_A \int \boldsymbol{\sigma} = -C_A M_{GT}. \end{aligned} \tag{65}$$

In general, all observables can be expressed in terms of the *form factor coefficients*, ${}^V F^{(k)}_{KLs}$ and ${}^A F^{(k)}_{KLs}$, and these are also the quantities which can be derived from experimental data. If one wishes to compute observables theoretically from some kind of model or to extract nuclear structure information from a β-decay experiment, the form factors need to be related to the nuclear transition matrix elements. In eq. (65) the particularly simple relation for Fermi and Gamow-Teller matrix elements are presented.

Since only $M_0(1,1)$ and $M_1(1,1)$ contribute significantly to allowed decays, very few of the particle parameters are non-negligible:

$$\begin{aligned} b^{(0)}_{00} &= L_0 M_0(1,1) M'_0(1,1) \\ b^{(0)}_{11} &= -\sqrt{3} L_0 M_1(1,1) M'_1(1,1) \\ b^{(1)}_{01} &= L_0 \Lambda_1 \frac{p_e}{W_e} \left[M_0(1,1) M'_1(1,1) + M_1(1,1) M'_0(1,1) \right] \\ b^{(1)}_{11} &= \sqrt{2} L_0 \Lambda_1 \frac{p_e}{W_e} M_1(1,1) M'_1(1,1) \end{aligned} \tag{66}$$

Of special importance here is that no terms with $k \geq 2$ appear. In normal decays the quoted particle parameters are sufficient to describe the observables to a very good approximation, however, if the allowed matrix elements are somehow suppressed, higher-order effects could become important, see for instance [6].

9. *Examples*

In this section we apply our formula for the correlation function to some examples in order to investigate the possible consequences of several intermediate states being populated in a cascade transition.

9.1 Allowed GT-transitions through a single intermediate state

This is probably the simplest case we can consider. We choose the hypothetical decay sequence

$$1^+ \to 0_2^+ \to 0_1^+.$$

Since the intermediate state has a well-defined parity only terms with even k can contribute, see the discussion following eq. (50). Therefore, we only need to consider $k = 0$ in the correlation function in eq. (54), a conclusion which is independent of our choice of I_B. This already leads to our first important result, namely that the directional distribution in an allowed β-delayed α decay is *isotropic*, because $P_0(\cos\theta)$ is a constant. This fact is a direct consequence of the leptons being emitted in an $L = 0$ wave, which is spherically symmetric.

Our problem directly dictates $\lambda_B = \lambda'_B$ and $I_B = I'_B = 0$. The selection rules in eq. (55) then provide us with the constraints $L'_2 = L_2 = I_B$ and $K'_1 = K_1 = 1$. The direct expression for the correlation function is

$$\begin{aligned} W &= 4\pi G_\beta^2 F_0 (-1)^{-1+0+0+0+1+1+0} \left[(2\cdot 0+1)(2\cdot 0+1)(2\cdot 0+1)\right]^{\frac{1}{2}} \\ &\quad \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} |\langle 0_1^+ || L=0 || 0_2^+ \rangle|^2 b_{11}^{(0)} P_0(\cos\theta_{\beta\alpha}) \\ &= 4\pi G_\beta^2 F_0 L_0 |C_A|^2 |M_{GT}(0_2^+)|^2 |\langle 0_1^+ || L=0 || 0_2^+ \rangle|^2. \end{aligned} \tag{67}$$

The result seems reasonable: We have the β decay part given by the β decay coupling constants, G_β and C_A, the Fermi function[6] $F(Z, W_e) = F_0 L_0$ and the squared Gamow-Teller matrix element, $|M_{GT}|^2$. Furthermore, the reduced matrix element for the α transition and a constant front factor of 4π appear.

If we do a similar calculation for a cascade with $I_B = 2$, we obtain instead

$$W = \frac{4\pi}{5} G_\beta^2 F_0 L_0 |C_A|^2 |M_{GT}(2_1^+)|^2 |\langle 0_1^+ || L=0 || 2_1^+ \rangle|^2. \tag{68}$$

The general structure is exactly the same as before, but the front factor changes by a factor $\frac{1}{5}$. That somehow bothers me, and I suspect there could be missing a factor of $(2I_B + 1)$ somewhere, or perhaps $[(2I_B + 1)(2I'_B + 1)]^{\frac{1}{2}}$.

[6] Apparently there is not complete consensus in the litterature about whether to denote F_0 or $F_0 L_0$ the *Fermi function*.

9.2 Allowed GT-transitions through several intermediate states

We now take a look at the cascade

$$1^+ \to 0_2^+, 0_3^+ \to 0_1^+,$$

i.e. a case where two intermediate states could potentially be populated. According to the selection rules in eq. (55) we have $k = 0$, $K_1' = K_1 = 1$ and $L_2' = L_2 = I_B = I_B' = 0$. Calculation of the correlation function gives us

$$\begin{aligned} W &= 4\pi G_\beta^2 F_0 L_0 |C_A|^2 \Big[|\langle 0_1^+ || L=0 || 0_2^+ \rangle|^2 |M_{GT}(0_2^+)|^2 \\ &\quad + \langle 0_1^+ || L=0 || 0_2^+ \rangle \langle 0_1^+ || L=0 || 0_3^+ \rangle^* M_{GT}(0_2^+) M_{GT}(0_3^+) \\ &\quad + \langle 0_1^+ || L=0 || 0_3^+ \rangle \langle 0_1^+ || L=0 || 0_2^+ \rangle^* M_{GT}(0_3^+) M_{GT}(0_2^+) \\ &\quad + |\langle 0_1^+ || L=0 || 0_3^+ \rangle|^2 |M_{GT}(0_3^+)|^2 \Big] \\ &= 4\pi G_\beta^2 F_0 L_0 |C_A|^2 \Big| \langle 0_1^+ || L=0 || 0_2^+ \rangle M_{GT}(0_2^+) + \langle 0_1^+ || L=0 || 0_3^+ \rangle M_{GT}(0_3^+) \Big|^2 . \end{aligned} \tag{69}$$

As in the previous example, the result appears rather simple in form, but it is worth noting that the contributions from the two intermediate states add up *coherently*. From a naïve point of view this could potentially complicate the analysis of experimental spectra or the theoretical prediction of spectra from nuclear models, since the reduced matrix elements could in principle be complex and carry a phase between 0 and 2π. In fact, the challenge is not quite so severe since, as we briefly touched upon earlier, the reduced matrix elements can be chosen real if the operator, which is responsible for the transition, is a Hamiltonian operator which commutes with the time reversal operator [3]. This means that we only need to determine the *magnitude* and *sign* of the matrix elements.

As our next example we take a cascade where the intermediate states have differing spins:

$$1^+ \to 0_2^+, 2_1^+ \to 0_1^+.$$

The selection rules in this case only allow two terms to contribute to the correlation function for allowed transitions, see Table 0.2. We obtain

$$\begin{aligned} W &= 4\pi G_\beta^2 F_0 L_0 |C_A|^2 \Bigg[\big[(2\cdot 0+1)(2\cdot 2+1)(2\cdot 2+1)\big]^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{Bmatrix} \\ &\quad \times \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} |\langle 0_1^+ || L=0 || 0_2^+ \rangle|^2 |M_{GT}(0_2^+)|^2 + \big[(2\cdot 0+1)(2\cdot 2+1)(2\cdot 2+1)\big]^{\frac{1}{2}} \\ &\quad \times \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 2 & 2 & 0 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{Bmatrix} |\langle 0_1^+ || L=2 || 2_1^+ \rangle|^2 |M_{GT}(2_1^+)|^2 \Bigg] \\ &= 4\pi G_\beta^2 F_0 L_0 |C_A|^2 \Big[|\langle 0_1^+ || L=0 || 0_2^+ \rangle|^2 |M_{GT}(0_2^+)|^2 \\ &\qquad + \frac{1}{5} |\langle 0_1^+ || L=2 || 2_1^+ \rangle|^2 |M_{GT}(2_1^+)|^2 \Big]. \end{aligned} \tag{70}$$

Again, a factor $\frac{1}{5}$ appear on the term related to the 2^+ intermediate state, possibly indicating an issue with the normalisation. The result in (70) is fundamentally different from the preceding result, since the contributions from the two intermediate states are added *incoherently*. This is simply the consequence of only considering allowed transitions, i.e. transitions with $k = 0$, since the selection rules then impose $I_B = I'_B$, making the cross-terms disappear. Were we to allow for $k = 2, 4$

Table 0.2: Quantum numbers for the two terms that contribute to the allowed transitions in the $1^+ \to 0^+_2, 2^+_1 \to 0^+_1$ cascade.

k	λ_B	λ'_B	L_2	L'_2	K_1	K'_1
0	0^+_2	0^+_2	0	0	1	1
0	2^+_1	2^+_1	2	2	1	1

also, some cross-terms might start to appear. These terms would contain first- or second-forbidden matrix-elements though, and we would not expect them to result in pronounced interference effects.

When we consider the correlation function in eq. (54) it has a feature, which we have not commented upon yet: The result appears as a sum over k with each term multiplied by a k'th order Legendre polynomial. In many experiments with β-delayed α breakups the direction of the β particle is not measured, and the proper correlation function would be obtained by integrating over $\theta_{\beta\alpha}$ and the energy of the β particle. Since $P_k(\cos\theta)$ integrate to zero except for $k = 0$, only terms with $k = 0$ contribute to the spectrum in this type of experiment.

10. Summary

We have generalised the existing theory of correlations in β-delayed α decays to allow for the possibility of mutiple intermediate taking part in the cascade. We have derived the correlation function and found an expression for the β-decay particle parameters which replaces the standard definition. The investigation of a few examples have led to the following conclusions:

1. If several intermediate states of identical angular momentum are populated in the β decay, their contributions to the correlation function add coherently.

2. If several intermediate states of different angular momentum are populated in an allowed β decay, their contributions to the correlation function add incoherently. Higher-order cross-terms could appear and cause (small) interference effects.

3. If several intermediate stated of different angular momentum are populated in the β decay, their contributions add incoherently if the direction of the β particle is not observed.

It it is important to realise that in the present treatment we have ignored the effects associated with the nuclear recoil. The nuclear recoil may induce correlations, particularly in the delayed α breakup of light nuclei, that are of the same order of magnitude as the effects caused by the forbidden matrix-elements. The standard reference on this subject seems to be [7].

A Matrix elements of representations of the rotation group

The matrix elements of the $(2L+1)$-dimensional irreducible representation of the rotation group, $\mathscr{D}^L$, have some basic properties, which are useful when doing explicit calculations. We only list these basic properties here; for a more in-depth discussion of the rotation operator and its representations, see for instance sec. 3.5 of [1] or sec. 3.2.1 of [3].

$$\mathscr{D}^L_{\mu M}(\mathbf{p}\to\mathbf{z}) = \mathscr{D}^{L*}_{M\mu}(\mathbf{z}\to\mathbf{p}) \tag{71}$$

$$\mathscr{D}^{L*}_{\mu M}(\mathbf{p}\to\mathbf{z}) = (-1)^{\mu-M}\mathscr{D}^L_{-\mu-M}(\mathbf{p}\to\mathbf{z}) \tag{72}$$

$$\mathscr{D}^L_{\mu M}(\mathbf{p}\to\mathbf{z}) = \sum_{M'}\mathscr{D}^L_{\mu M'}(\mathbf{p}\to\mathbf{z}')\mathscr{D}^L_{M'M}(\mathbf{z}'\to\mathbf{z}) \tag{73}$$

Here, the argument $(\mathbf{z}\to\mathbf{p})$ denotes the three Euler angles, (ϕ,θ,γ), while $(\mathbf{p}\to\mathbf{z})$ represents the inverse rotation. A product of two matrix elements can be expressed as a sum over single matrix elements:

$$\mathscr{D}^L_{\mu M}(\mathbf{p}\to\mathbf{z})\mathscr{D}^{L'}_{\mu' M'}(\mathbf{p}\to\mathbf{z}) = \sum_{k=|L-L'|}^{L+L'}\langle LL'\mu\mu'|k\tau\rangle\mathscr{D}^k_{\tau N}(\mathbf{p}\to\mathbf{z}), \tag{74}$$

where $\tau=\mu+\mu'$ and $N=M+M'$. Finally, some special matrix elements are directly related to the spherical harmonics and Legendre polynomials:

$$\mathscr{D}^L_{\mu 0}(\mathbf{z}\to\mathbf{p}) = \left(\frac{4\pi}{2L+1}\right)^{\frac{1}{2}}Y_L^{\mu *}(\theta,\phi) \tag{75}$$

$$\mathscr{D}^L_{0M}(\mathbf{z}\to\mathbf{p}) = \left(\frac{4\pi}{2L+1}\right)^{\frac{1}{2}}Y_L^{-M}(\theta,\phi) \tag{76}$$

$$\mathscr{D}^L_{00}(\mathbf{z}\to\mathbf{p}) = P_L(\cos\theta) \tag{77}$$

B Properties of the Wigner-symbols

When dealing with the coupling of angular momenta most physicists have been exposed to the Clebsch-Gordan coefficients, $\langle j_1 j_2 m_1 m_2|j_3 m_3\rangle$. The Wigner $3j$-symbols are related to the Clebsch-Gordan coefficients through the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-\frac{1}{2}}\langle j_1 j_2 m_1 m_2|j_3 - m_3\rangle. \tag{78}$$

The $3j$-symbols are not affected by an even permutation of the columns, but an odd permutation, or the transformation of all m to $-m$, corresponds to a multiplication by $(-1)^{j_1+j_2+j_3}$, i.e.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \tag{79}$$

$$(-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \cdots \tag{80}$$

$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{81}$$

The $3j$-symbols satisfy the following orthogonality relations:

$$\sum_{j_3 m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{82}$$

$$\sum_{m_1 m_2} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \delta_{j_3 j_3'} \delta_{m_3 m_3'} \tag{83}$$

Sums over multiple $3j$-symbols containing the same quantum numbers can often be carried out using a $6j$-symbol:

$$\sum_{m_4 m_5 m_6} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & m_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & m_2 & m_6 \end{pmatrix}$$
$$\times \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{84}$$

The $6j$-symbol has very simple symmetry properties: It is invariant under the permutation of any two columns, and it is invariant under the interchange of the upper and lower arguments an any two columns, i.e.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ j_5 & j_6 & j_4 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_4 & j_2 & j_3 \end{Bmatrix} = \cdots \tag{85}$$

Furthermore, the $6j$-symbol is zero unless j_1, j_2 and j_3 satisfy the triangle condition, i.e.

$$|j_2 - j_3| \leq j_1 \leq j_2 + j_3. \tag{86}$$

Note that through the allowed permutations of the arguments in the $6j$-symbol, the triangle condition can be applied to several combinations of the j's.

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