Sequential three-body decays

Sekventielle trepartikel-henfald

Albert Freud Natorp

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Abstract

In this thesis, we construct and investigate a sequential model of three-body nuclear decays using R-matrix theory. Since R-matrix theory is originally designed for two-particle channels only, we have attempted to combine two R-matrix expressions to calculate an amplitude for a sequence of two-body decays. The model is compared to earlier sequential models: specifically the one by Lane & Thomas in [8], and the one by Balamuth et al. in [5]. Also, the effects of identical particle symmetrization are investigated. The model is shown to produce a dramatic but unphysical symmetrization effect in the case of ^{9}Li decays to $\alpha + \alpha + n$, and we discuss how this may be an artifact of the completely delocalized plane waves used in the construction of the model. A solution is proposed by considering the effects of more localized decay fragments.

Contents

1 Introduction

This thesis investigates nuclear β-decays followed by fragmentation into three bodies. Schematically, we might write:

$$i \underset{\beta}{\rightarrow} A \rightarrow a + b + c \tag{1}$$

where i represents the initial nucleus, which β-decays to the short-lived nucleus A, which then in turn breaks apart into the three fragments a, b and c. We would like to make a model that describes this kind of process, so that it may be used in analysis of experiments.

The (very idealized) experiment we imagine in the development of our model of three-body decays measure the momenta of all three final state fragments - both magnitude and direction. A stylized representation of an experiment like this has three detectors - one measuring each of the particles. A drawing of this idealized experiment is seen in figure 1.

Three-body problems are notoriously hard, and it is not the aim of this thesis to present a full three-body treatment of these types of decays. Instead we shall investigate a subset of processes: namely those that can be modeled by *sequences of two-body decays*. We shall assume that one of the fragments a, b or c splits off of the product of the β-decay (nucleus A in equation 1) first, leaving a *recoil nucleus* (let's call it B). Since this "first fragment" has split off before the others we label it 1:

$$i \underset{\beta}{\rightarrow} A \rightarrow 1 + B \tag{2}$$

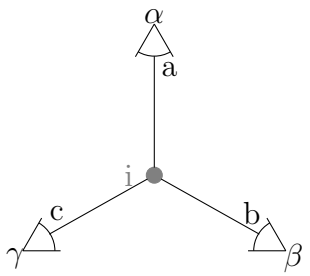

Figure 1: A stylized drawing of the experiment. A nucleus i decays to three particles labeled a, b and c. Three detectors, labeled α, β and γ measure the momenta of the three particles.

After this two-body process the fragments 1 and B move away from each other, before the recoil B breaks apart resulting in the remaining two fragments (let's label them 2 and 3 since they are emitted after 1):

$$i \underset{\beta}{\rightarrow} A \rightarrow 1 + B \rightarrow 1 + 2 + 3 \tag{3}$$

Since we don't observe the recoil nucleus B we don't know which of the fragments was emitted first - hence we must sum the amplitudes of the three sequential decay paths. This sequential model allows us to adapt existing *two-body models* to this special class of three-body decays, as an approximation. Specifically we shall build our model on the R-matrix theory introduced by Wigner and Eisenbud in 1947 (see e.g. [13]).

Since some of the topics used in developing our model are not considered a part of the standard curriculum in a Master's Degree of physics, we shall include some theoretical prerequisites:

- Section 2 is a brief reminder of relativistic kinematics, including some important results for three-body decays, such as the Dalitz plot.
- Section 3 is a (very) brief discussion of the quantum mechanical three-body problem, and it mainly serves to introduce the *Jacobi coordinates*, which are ubiquitous in describing three-body systems.
- Section 4 is an overview of the general ideas and central results of R-matrix theory needed for the development of our model.

These sections of theoretical prerequisites are followed by section 5, in which our model is developed and presented. The results of section 3 are used to build a description of the problem, while the results of section 4 are used to derive an expression for the amplitude of a sequential decay. The development of the model is of a phenomenological nature - it is not "real theory" build on our most fundamental understanding of reality, but instead it tries to adapt existing theory to a broader range of phenomena, based partly on existing models and partly on experiment. For instance, the final model contains several parameters that must be fitted to experiment.

The motivation to develop this model comes from a couple of nuclear processes of the form described in equation 1, that are being studied experimentally. The triple-α decay of excited states in ^{12}C have been studied

extensively because of its importance in understanding stellar fusion beyond ^{4}He (see e.g. [2]). Another process is the β-decay of ^{9}Li:

$$^9\mathrm{Li} \underset{\beta}{\to} {}^9\mathrm{Be} \to \alpha + \alpha + n \tag{4}$$

Lastly, experiments at the ISOLDE facility at CERN on the decay of ^{8}He resulting in α, t (triton) and n are being conducted as well. All of these experiments would be interesting to analyze using a sequential model of 3-body decays.

Other sequential models of three body decays have been proposed in the past. We will focus on two notable contributions: One treatment is found in the important review article on R-matrix theory by Lane and Thomas [8], focusing on a simplified case of isolated levels in both decays. We show that our model can reproduce their result in section 7, and that our work is in fact a quite natural generalization of their approach.

The other contribution is specifically on the triple-α decay of ^{12}C and is found in an article by Balamuth et al. [5]. The case of ^{12}C is special in several ways: 1) The final fragments have 0 spin, making the angular momentum coupling less troublesome, and 2) since the final fragments are identical particles the amplitude must be symmetric under interchange of any two fragments. Balamuth et al. are not very precise in this distinction between interfering sequential decay paths and symmetrization effects due to identical particles. In section 8 we investigate the consequences of identical particles in our model, and in section 9 we compare the work of Balamuth et al. to our model.

2 Kinematics of sequential three body decays

The following section is a brief reminder of the kinematics of two- and three-body decays, as well as some specific results arising from sequential decays. It is largely based on [7], and we use the $(+,-,-,-)$ metric.

If a mass M at rest decays into two fragments, 1 and B say, we may express the conservation of 4-momentum as:

$$(M, 0) = (E_1, \mathbf{p}_1) + (E_B, \mathbf{p}_B) \tag{5}$$

where the notation $(E, \mathbf{p})$ refers to a four momentum with energy component E, and spatial components $\mathbf{p}$. From this we may calculate the energy of

one of the fragments, by evaluating squares of 4-momenta (remember these are independent of the chosen frame of reference). E.g. if we isolate the 4-momentum of particle B in equation 5 we have $((M,0)-(E_1,\mathbf{p}_1))^2 = ((E_B,\mathbf{p}_B))^2 = m_B^2$, and we can isolate the energy of particle 1:

$$E_1 = \frac{M^2 + m_1^2 - m_B^2}{2M} \tag{6}$$

For a sequential three-body decay $A \to 1 + B \to 1 + 2 + 3$ we don't know the mass of the recoil fragment B, but we do know the energy. Then we can instead calculate the mass of the recoil from the energy by inverting the above expression:

$$m_B = \sqrt{M^2 + m_1^2 - 2ME_1} \tag{7}$$

For a mass M at rest decaying to three bodies the conservation of four-momentum means that all linear momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ lie in a plane (the vectors must form a closed triangle since they sum to zero). Furthermore, with a total of $3 \times 3 = 9$ degrees of freedom, and with four of them specified by the conservation of four momentum, we have $9 - 4 = 5$ degrees of freedom left. These may be chosen as follows: We may specify two of the final energies, E_1, E_2, since the third is then given from energy conservation. Then we may specify the direction of the first fragment by two angles, (θ_1, ϕ_1) and then the last degree of freedom will be used to specify the rotation of the plane of momenta around the axis of the first momentum $\mathbf{p}_1$. If a z-axis is chosen before the decay happens, the three angles correspond to the Euler angles rotating the initial z-axis to $\mathbf{p}_1$.

Another important aspect of the three-body decay is that the conservation of 4-momentum puts non-trivial limits on the allowed choices of the two "free energies", say E_1 and E_3. One of the energies may be chosen freely within certain limits. For instance, the energy of fragment 3 could be chosen between the case where the fragment is stationary, $E_3 = m_3$, and the other case where it has all the kinetic energy of the system: $E_3 = M - m_1 - m_2$. But, the limits on the second energy, E_1, depend on this choice. To see this we change to a new set of variables. If the four momentum of particle i is denoted by p_i, and the total four-momentum of the three particle system is denoted by P we define the new quantity m_{ij}^2 as:

$$m_{ij}^2 = (p_i + p_j)^2 = (P - p_k)^2 = M^2 + m_k^2 - 2ME_k. \tag{8}$$

m_{ij}^2 is the norm-square of the four momentum of the composite particle consisting of i and j. This quantity turns out to be easier to work with, and we note that it is linearly dependent on the energy of one of the fragments (with a negative slope; large values of m_{12}^2 means small values of E_3 and vice versa)[1]. The limits on E_3 just put forth are now limits on m_{12}^2 instead:

$$(m_1 + m_2)^2 \leq m_{12}^2 \leq (M - m_3)^2 \tag{9}$$

and E_1 is now translated to:

$$m_{23}^2 = (p_2 + p_3)^2 = m_2^2 + m_3^2 + 2E_2E_3(1 - \mathbf{p}_2 \cdot \mathbf{p}_3/(E_2E_3))$$

The limits on this expression are seen to occur when the linear momenta of particle 2 and 3 are parallel and antiparallel, and the exact values depend on E_3, as claimed. The limits can be calculated by changing to the system where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. We shall not perform the calculation here, but simply quote the result from [7]:

$$(m_{23})_{\max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2 \tag{10}$$

$$(m_{23})_{\min} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2 \tag{11}$$

with

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}} \tag{12}$$

$$E_3^* = \frac{M^2 - m_{12}^2 - m_3^2}{2m_{12}} \tag{13}$$

Two examples of these allowed regions are plottet in figure 2.

These plots of m_{12}^2 against m_{23}^2 are not just interesting because of the allowed region of energies. They have another quite interesting property. Let us write out the partial decay rate for a three body decay $d\Gamma$ as given in [7]:

$$d\Gamma = \frac{1}{(2\pi)^5}\frac{1}{16M}|\mathcal{M}|^2 dE_1 dE_3 d\alpha d(\cos(\beta)) d\gamma \tag{14}$$

[1]This is not surprising; the more four-momentum the composite fragment of i and j have, the less is left for the last fragment k.

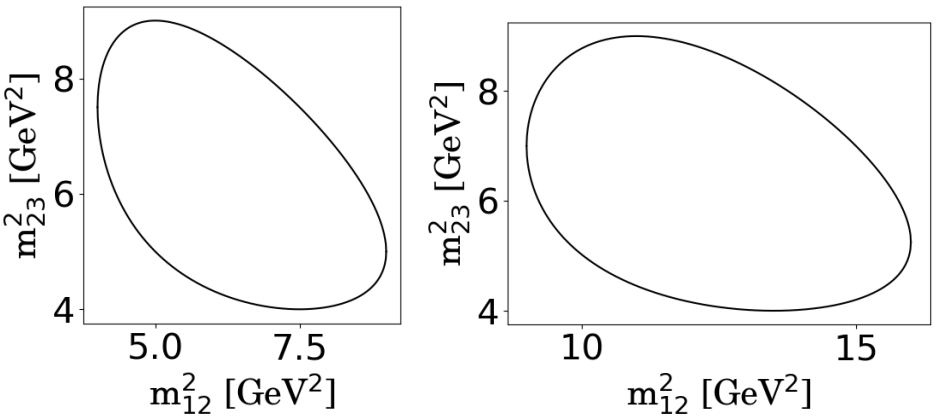

Figure 2: Two examples of the allowed regions of m_{ij}^2, corresponding to allowed regions of energies of two of the fragments. The plot on the left corresponds to equal masses m_3 (the horizontal axis) and m_1 (the vertical axis) and gives a symmetric region, while the plot on the right has $m_1 = 2m_3$ resulting in an asymmetric region. When events are plotted on axes like these the resulting plot is called a Dalitz plot.

where α, β and γ represent the Euler angles, and $\mathcal{M}$ is the matrix element of the process. If we average over the spin states of the decaying particle, there are no preferred Euler angles (the partial rate is independent of the orientation of the plane of linear momenta when the initial state has full rotation symmetry) and the integration over them becomes trivial. We get:

$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{8M}|\mathcal{M}|^2 dE_1 dE_3 = \frac{1}{(2\pi)^3}\frac{1}{32M^3}|\mathcal{M}|^2 dm_{12}^2 dm_{23}^2 \qquad (15)$$

where the last equality follows from the definition of m_{ij}^2. This means that *if* the matrix element $\mathcal{M}$ does not depend on the energies of the fragments (the point in the allowed region), we will see a uniform distribution of events over the entire region - in other words: any structure in the plot must be structure in the matrix element. This kind of plot is called a *Dalitz plot* after R. H. Dalitz who introduced this technique for studying matrix elements in three-body processes.

It is also worth mentioning that for a given choice of m_{12}^2 - i.e. a given vertical line in the Dalitz plot - the different values of m_{23}^2 correspond to different angles between the two fragments 3 and 1. That is, the Dalitz plot contain information on the angular correlation between fragments.

3 Three-body quantum mechanics

Since we will be dealing a whole lot with three particles, we will briefly sketch some fundamental three-body quantum mechanics to familiarize the reader with central concepts and notation.

Let us consider three particles a, b and c, with masses m_a, m_b and m_c. Their positions are described by vectors $\mathbf{r}_a$, $\mathbf{r}_b$ and $\mathbf{r}_c$ respectively (see figure 3 on the left), with corresponding momenta $\mathbf{p}_a$, $\mathbf{p}_b$ and $\mathbf{p}_c$. They interact with each other through a potential $V(\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c)$. The Hamiltonian of this system is:

$$H = \frac{\mathbf{p}_a^2}{2m_a} + \frac{\mathbf{p}_b^2}{2m_b} + \frac{\mathbf{p}_c^2}{2m_c} + V(\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c) \tag{16}$$

We remind the reader that in the *quantum mechanical two-body problem* the next step would be to perform a coordinate transformation that separates the center of mass movement from the relative movement of the two particles. A similar thing can be done here. However, in the two-body problem the coordinate transformation effectively leaves us with a single particle described by the relative coordinate and the reduced mass. This is not the case in the three-body system as is evident from a quick count of the degrees of freedom: $3 \times 3 = 9$ total, and reserving 3 for the center of mass we are left with 6 internal degrees of freedom - or *two* relative position vectors. Furthermore; in the two-body problem the choice of relative position vector is fairly obvious and almost unambiguous (the only choice we have to make is the direction of the relative position vector), while there are several equally good options for choosing the relative position vectors in the three-body problem, as we shall see shortly.

The usual approach is to define a subsystem of two of the particles within the three-body system. Let us call this subsystem B. Then we can describe the lone particle relative to the subsystem (effectively a two-body system), and the two particles in the subsystem relative to each other (another two body system). Let us write out the coordinates if we (arbitrarily) choose the

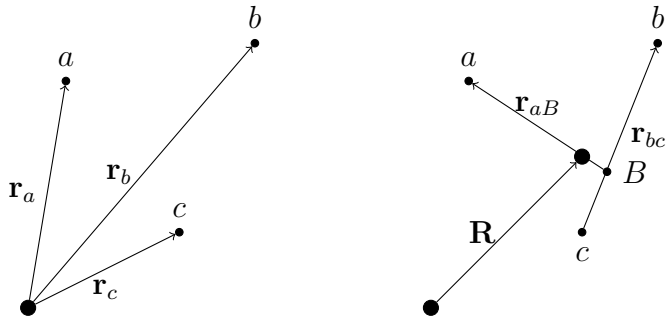

Figure 3: The coordinates where all positions are tracked relative to the origin are shown on the left, the relative coordinates of equations 17, 18 and the center of mass coordinate in equation 19 are shown on the right. The coordinates on the right is *one possible set* of coordinates. We might as well have chosen the subsystem B to consist of particles a and c, giving us two new relative position vectors, or any other permutation of abc in equations 17 and 18.

subsystem B to consist of particles b and c:

$$\mathbf{r}_{bc} = \mathbf{r}_b - \mathbf{r}_c \tag{17}$$

$$\mathbf{r}_{aB} = \mathbf{r}_a - \frac{m_b\mathbf{r}_b + m_c\mathbf{r}_c}{m_b + m_c} \equiv \mathbf{r}_a - \mathbf{r}_B \tag{18}$$

$$\mathbf{R} = \frac{m_a\mathbf{r}_a + m_b\mathbf{r}_b + m_c\mathbf{r}_c}{m_a + m_b + m_c} \tag{19}$$

Here $\mathbf{r}_{bc}$ is the relative coordinate of the two particles b and c, $\mathbf{r}_B$ is the center of mass coordinate of the $b + c$ subsystem, $\mathbf{r}_{aB}$ is the relative coordinate of particle a and the subsystem B, and $\mathbf{R}$ is the center of mass coordinate of the entire three-body system [12]. These coordinates are often referred to as *Jacobi coordinates*. They are plotted on the right in figure 3. To progress further we must find the canonical momenta so that we can write out our Hamiltonian in our new Jacobi coordinates. We shall simply state the result here referring to [12]:

$$\mathbf{p}_{bc} = \mu_{bc}\left(\mathbf{p}_b/m_b - \mathbf{p}_c/m_c\right) \tag{20}$$

$$\mathbf{p}_{aB} = \mu_{aB}\left(\mathbf{p}_a/m_a - \mathbf{p}_B/m_B\right) \tag{21}$$

$$\mathbf{P} = \mathbf{p}_a + \mathbf{p}_b + \mathbf{p}_c \tag{22}$$

where $\mu_{bc} = \frac{m_b m_c}{m_b + m_c}$ is the reduced mass of particles b and c (and analogously for μ_{aB}), $\mathbf{p}_B = \mathbf{p}_b + \mathbf{p}_c$ is the momentum of the subsystem B (consisting of $b + c$), and $m_B = m_b + m_c$ is the mass of the subsystem B (still $b + c$ in this case). Inverting these momentum relations and inserting them in the Hamiltonian (equation 16) one can show that [12]:

$$H = \frac{\mathbf{P}^2}{2(m_a + m_b + m_c)} + \frac{\mathbf{p}_{bc}^2}{2\mu_{bc}} + \frac{\mathbf{p}_{aB}^2}{2\mu_{aB}} + V(\mathbf{R}, \mathbf{r}_{aB}, \mathbf{r}_{bc}) \tag{23}$$

Assuming that the potential does not depend on the total center of mass coordinate, we see that the center-of-mass motion separates from the internal coordinates, as in the two-body case.

Of course the choice of particle a as the lone particle and $b + c$ as the subsystem is completely arbitrary, and we might as well have chosen any permutation of a, b and c in equations (17-22) to obtain a valid separable Hamiltonian. In fact, a certain choice of coordinate transformation might make specific calculations easier: for instance if we want to calculate an amplitude where particle a breaks off leaving the remaining two particles b and c in a (temporarily) bound system, the coordinates appearing in equations (17-22) would be much better than any other set because the subsystem appears explicitly in these coordinates.

A complete solution of the quantum mechanical three-body problem would then proceed from here: in the case of scattering one might write out integral equations for the scattering states (analogous to the Lippmann-Schwinger equation), or in other ways try to generalize the methods of two-body scattering (see [12] for a detailed discussion). However, we shall not attempt a complete three-body treatment - instead we shall investigate the possibility of sequential two-body processes in this thesis and so we stop our discussion of the quantum mechanical three-body problem here.

4 A brief introduction to R-matrix theory

The theory development in this thesis builds on the R-matrix framework by Wigner and Eisenbud. To increase the readability of the thesis we shall give a brief introduction to R-matrix theory, based on chapter 16 in [10] and the seminal review article on R-matrix theory by Lane and Thomas [8]. The following section is by no means a complete development of the theory - it is merely a summary containing the most important features, and most results

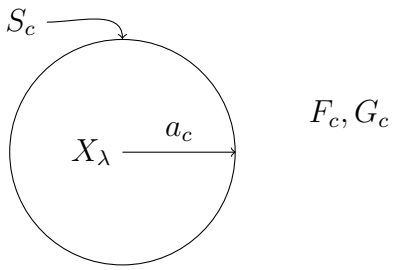

Figure 4: An illustration of the two regions of physical space in R-matrix theory. The circle represents the surface of the division, the channel surface S_c. It has a radius a_c, and inside of this circle the Schrödinger equation contains nuclear forces and has solutions X_λ. Outside this region, the two particles are too far away from each other for the short-range nuclear forces, and the solutions to the Schrödinger equation in this region are the Coulomb wave functions F_c and G_c.

are simply stated, and not derived. The interested reader is referred to the review article [8] for further details.

4.1 The broad picture of R-matrix theory

R-matrix theory is designed to calculate amplitudes of reactions and scattering experiments where two nuclei $1 + 2$ enter in a channel c, interact, and continue out as two (possibly) different nuclei $3 + 4$ in a (possibly) different channel c'. In bra-ket notation, we could imagine coupling these different channels through an operator T:

$$\mathcal{A}_{c',c} = \langle c'|T|c\rangle. \tag{24}$$

The R-matrix approach to calculating this amplitude is to split physical space into two distinct regions: one where the nuclei do not interact (other than through Coulomb forces), and one where they do. This split of the region is purely a theoretical trick. As R-matrix theory is formulated in terms of wave functions, the consequence of this split is that we solve the Schrödinger equation in two separate domains: The outside region beyond the range of nuclear forces, and the inside region where nuclear forces are present. See figure 4 for an illustration.

The outer region where no nuclear forces are present is simply a two-body Coulomb problem - a problem that has been solved, with solutions in

the form of Coulomb wave functions F (the one regular at the origin) and G (the irregular one) for a continuum of energies E. These solutions are of course wave functions in the relative coordinate between the two particles. The known solutions to this problem is one of the reasons we perform the split of physical space. We shall use the following notation for the Coulomb wave functions:

$$F_l(kr, \eta), \quad G_l(kr, \eta). \tag{25}$$

I.e. the Coulomb functions depend on the relative orbital angular momentum between the two fragments, l, and the Sommerfeld parameter $\eta = \alpha Z_1 Z_2 \sqrt{\mu c^2 / 2K}$, where α is the fine structure constant, Z_i are the nuclear charge numbers of the two nuclei, μ is the reduced mass of the two nuclei, c is the speed of light, and K is the kinetic energy of the two nuclei. Their argument is the dimensionless distance $k \cdot r$ which is a product of the wavenumber k and the radial distance r. The wavenumber is related to the kinetic energy (for positive energy channels) as:

$$k = \frac{\sqrt{2\mu K}}{\hbar}. \tag{26}$$

The regular and irregular solutions are combined to form incoming ($\mathcal{I}_c$) and outgoing ($\mathcal{O}_c$) wave functions for all channels c:

$$\mathcal{I}_c = (G_c - iF_c)\, e^{i\omega_c} \tag{27}$$

$$\mathcal{O}_c = (G_c + iF_c)\, e^{-i\omega_c} \tag{28}$$

where ω_c is the Coulomb phase shift, given by:

$$\omega_c = \sum_{n=1}^{l} \arctan \frac{\eta_c}{n}, \quad \omega_c = 0 \quad \text{for} \quad l = 0. \tag{29}$$

The total external wave function of the scattering problem can be written as a superposition of $\mathcal{I}_c$ and $\mathcal{O}_c$ for all channels c:

$$\Psi = \sum_c (x_c \mathcal{O}_c + y_c \mathcal{I}_c). \tag{30}$$

Solving the scattering problem comes down to relating the coefficients of the incoming waves (i.e. the incoming channels which we control in an experiment), to the coefficients of the outgoing waves (i.e. the outgoing channels

we measure as the result of the experiment). A collision matrix U containing this relation between the incoming and outgoing coefficients is defined implicitly by the following relation:

$$x_c = -\sum_{c'} U_{c,c'} y_{c'}. \tag{31}$$

The collision matrix must of course depend on the physics in the inner region where the scattering actually happens. Here nuclear forces enter the Schrödinger equation resulting in an unsolvable problem (at least for most nuclei). The beauty of R-matrix theory is that instead of solving the internal nuclear problem (which generally we can't), we attempt to work out as much of the energy dependence as we can without explicit reference to the details of the inner system, leaving us with a few energy independent parameters that can be fitted to experiment. Let us elaborate on the specifics.

The introduction of a boundary on the inner region means we know there is a set of discrete states[2]. These states are called the internal levels λ (with wave functions X_λ), with energies E_λ. In principle an infinite number of internal levels should be used, but in practice this is luckily not necessary because only the levels close to the specific energy contribute. A channel (i.e. a solution in the outer region) can couple into the internal levels "through" the boundary surface. The interaction radius might depend on the channel, hence each channel has its own boundary surface defined by the channel radius a_c. The channel radii a_c can be chosen freely as long as they are larger than the range of the nuclear potentials - otherwise the outside region would not be a Coulomb problem. The division of physical space is somewhat artificial, and most quantities *within* the theory depend on the exact value of this channel radius, but their dependence cancel out in the end leaving observable quantities (such as cross-sections) independent of this (somewhat) arbitrary choice. For example, the evaluation of the (outer) Coulomb wave funtions on the channel surface depend on the channel radius a_c. Echoing the notation in equation 24 where c and c' refer to external states (channels), the strategy of R-matrix theory is to represent the coupling between channels not through an operator T, but instead through the set of internal states λ:

$$\mathcal{A}_{c',c} = \sum_\lambda \langle c'|\lambda\rangle\langle\lambda|c\rangle \tag{32}$$

[2]The matching procedure on the boundary surface puts restrictions on the eigenvalues, as is seen e.g. in the infinite square well, making the spectrum discrete.

We must stress that equation 32 is dangerous in its simplicity. The channel wave functions and the internal wave functions are not defined in the same domain, and the bra-ket notation hides this very important fact. Hence, this expressions is only included to guide the intuition of the reader, and it is not to be understood as a rigorous result or definition.

To solve the scattering problem we must match the internal wave functions to the external wave functions. This is usually done by matching the logarithmic derivatives, since this matching ensures that both the wave function and its derivative is continuous across the boundary surface. Since we do not know the internal wave functions we parametrize their logarithmic derivatives and their value on the surface. Specifically, the value of the wave function on the surface is denoted by $\gamma_{\lambda,c}$, and is referred to as *the reduced width*. More precisely the *value of the internal wave function on the surface* S, called $\gamma_{\lambda,c}$, can be expressed as the overlap between the internal state λ (with wave function X_λ), and the external channel c (with "surface wave function"[3] ϕ_c) *on the boundary surface*:

$$\gamma_{\lambda,c} = \frac{\hbar}{\sqrt{2\mu_c a_c}} \int \phi_c X_\lambda dS \tag{33}$$

This integral representation of the value of X_λ on the surface is not obvious, and we encourage the reader to remember that the reduced width $\gamma_{\lambda,c}$ is the value of the internal wave function on the surface.

The value of the logarithmic derivative of the internal state wave function X_λ on the channel surface of channel c is defined as a parameter B_c. This definition locks the value of the actual derivative of the internal wave function on the surface (usually called $\delta_{\lambda,c}$). B_c must be the same for all internal levels[4]. Just like the channel radius a_c could be chosen freely (but not too small), so we can choose the value of the logarithmic derivative on the surface freely for each channel. Of course, the values of quantities inside the theory depend on this choice, but the dependencies cancel out exactly in

[3]The quantity is the non-radial part of the channel wave function. For a rigorous mathematical definition we refer to [8], page 270

[4]To solve the Schrödinger equation we must choose a boundary condition, and for any choice of boundary condition we get the whole set of internal levels. It is then not surprising that all internal levels must share the same boundary condition. And, since the different channels don't define the same inner regions (the channel radii are not all the same) it is also not surprising that the different channels can have different boundary conditions.

the expressions for measurable quantities. The logarithmic derivatives of the external (channel) wave functions pose no problem, as these wave functions are known.

4.2 Collision matrix

The collision matrix U must of course depend on the inner region. It turns out that the combination of reduced widths $\gamma_{\lambda,c}$, boundary condition parameters B_c and channel radii a_c along with the energies E_λ are in fact sufficient to determine the collision matrix unambiguously. The collision matrix can be expressed explicitly in terms of these quantities. We shall not derive such an expression here, but simply state one particular form suited for our purposes (see [6] eq. 2.43):

$$U_{c',c} = \Omega_{c'}\Omega_c \left[\delta_{c',c} + 2iP_{c'}^{1/2} P_c^{1/2} \sum_{\lambda,\mu} \gamma_{c',\lambda} A_{\lambda,\mu} \gamma_{\mu,c} \right]. \tag{34}$$

Here the matrix $A_{\lambda,\mu}$ is called the level matrix. It is a symmetric matrix in the internal levels, and it can be calculated from its inverse:

$$(A^{-1})_{\lambda,\mu} = \delta_{\lambda,\mu} \times (E_\lambda - E) - \sum_c \gamma_{\lambda,c}\gamma_{\mu,c}(S_c - B_c + iP_c) \tag{35}$$

where S_c is called the shift function and P_c is called the penetrability[5]. They are given in terms of the Coulomb wave functions by (dots denote differentiation with respect to the dimensionless argument):

$$S_c = k_c a_c \frac{F_c\dot{F}_c + G_c\dot{G}_c}{F_c^2 + G_c^2} \tag{36}$$

$$P_c = \frac{k_c a_c}{F_c^2 + G_c^2} \tag{37}$$

The Ω's are phases containing contributions from the introduction of a channel radius (the so-called "hard sphere scattering phase shift") as well as from Coulomb scattering. It can be calculated as:

$$\Omega_c = \exp\left(i(\omega_c - \phi_c)\right) \tag{38}$$

[5]They are in fact the real and imaginary parts of the logarithmic derivate of the external channel wave function.

where ω_c is the Coulomb phase shift we already met in the expression for the incoming and outgoing waves, and ϕ_c is the hard sphere scattering phase shift given by:

$$\phi_c = \arctan \frac{F_c}{G_c} \tag{39}$$

R-matrix theory has at this point parametrized the collision matrix in terms of the internal energies E_λ, the overlap integrals $\gamma_{\lambda,c}$, the boundary condition parameter B_c and the channel radius a_c. The former two quantities are not known, and must be either calculated or inferred from experiment[6], the latter two are essentially chosen freely - however it is (usually) most convenient to choose the radius and boundary condition as follows:

$$a_c = r_0 \left(A_1^{1/3} + A_2^{1/3} \right) \tag{40}$$

$$B_c = S_c(E = E_{\lambda_0}) \tag{41}$$

where E_{λ_0} is the lowest eigen-energy of the internal states and r_0 is a distance scale. It is often chosen as ~ 1.5 fm. Since E_λ and $\gamma_{\lambda,c}$ must be fitted from experiment the model can not make predictions "on its own" - but it does put constraints on the energy dependence of the cross section, and this can be tested through a fit to experiment.

For our purposes it is useful to rewrite equation 34 as a matrix equation in the following way (ignoring the extra diagonal contribution, which is okay in our case, since we will not investigate processes that end in the initial state):

$$\mathbf{U} = i\mathbf{\Omega_f}\sqrt{2\mathbf{P_f}}\gamma_{f,\lambda}\mathbf{A}\gamma_{\lambda,i}\mathbf{\Omega_i}\sqrt{2\mathbf{P_i}} \tag{42}$$

Please note that the reduced widths γ appearing in this equation are *matrices* - their indices don't refer to entries, but to the fact that they couple different sets of states: e.g. the reduced width *matrix* $\gamma_{f,\lambda}$ couple the set of states λ and the set of final states f.

At this point we shall include a comment on the interpretation of the collision matrix expression: the right most part $\gamma_i\mathbf{\Omega_i}\sqrt{2\mathbf{P_i}}$ is a coupling from the initial states to the internal R-matrix levels, containing a phase change Ω, a penetrability P and an overlap integral γ. Then the level matrix A

[6]Note that both the energies and overlap integrals in the theory depend on the particular choice of boundary condition parameter B_c and channel radius a_c, because the *observed* energies and widths depend on all four parameters.

mixes the internal states (corresponding essentially to a broadening of these states, since they couple to states in the continuum), followed by a coupling from the internal states to the final states $\mathbf{\Omega_f}\sqrt{2\mathbf{P_f}}\gamma_f$.

It still remains to relate the collision matrix to the amplitude. In general the relation between the collision matrix and the amplitude is complicated in R-matrix theory. However, in our specific case the relation is quite simple. We would like to calculate the R-matrix amplitude between states $i = (S_i, m_{S_i})$ and $c = (S_1, m_{S_1}, S_2, m_{S_2}, l, m_l)$, where the states i populate the internal levels $\lambda = (S_\lambda, m_{S_\lambda})$ through a Gamow-Teller dominated β-decay. In the special case we have here, where the initial channel is a single particle channel, and where the final channel is a state of definite angular momentum l, m_l we have:

$$\mathcal{A}_{c,i} = -i\sqrt{\pi}U_{c,i} \tag{43}$$

Since the initial state is a single particle state populating the λ-states via a β-decay we omit the k_i appearing in [8]. This result follows from [8], chapter VIII, equation 1.10, when our specific case is considered. We will not show it here.

This concludes the discussion of standard R-matrix theory. The rest of this section introduces some extensions that are necessary in the development of our model.

4.3 R-matrix formalism for β-decays

Since our initial state is not a two particle state it is not treatable in the standard formulation of R-matrix theory. Luckily the theory can be generalized to include this case. We shall not derive this here, but simply quote the result from [6], page 21, which is based on the photon-channel section in chapter XIII in [8]. The relevant substitution is in the coupling from the initial state to the internal states:

$$\gamma_{A,i}\mathbf{\Omega_i}\sqrt{2\mathbf{P_i}} \to \sqrt{f}g_{A,i} \tag{44}$$

where f is the integrated Fermi-function (a phase space factor), and $g_{A,i}$ is the β-strength from the initial state to the internal states [6].

The collision matrix for a process initiated by a β-decay is given in [6] on page 22 as:

$$U_{c,i} = \Omega_c\sqrt{2P_c}\sum_{\lambda,\lambda'}\gamma_{c,\lambda}A_{\lambda,\lambda'}g_{\lambda',i}\sqrt{f} \tag{45}$$

In matrix notation, writing out the collision matrix as a vector in final channels c:

$$\vec{U}_{c,i} = \mathbf{\Omega}_c \sqrt{2\mathbf{P}_c} \gamma_{c,\lambda} \mathbf{A} \vec{g}_i \sqrt{f} \tag{46}$$

where $\gamma_{c,\lambda}$ is now understood to be a matrix. This is analogous to the expression in equation 42.

It is worth noting that for scattering and reaction experiments we are interested in the cross-section, while for decays we want to calculate a rate. The substitution above almost corrects this automatically, except for a normalization constant as described in [6], p. 22. We shall not delve into this here because it amounts to a scaling and does not alter the energy dependence, but note that the procedure is necessary to obtain the correct rate.

4.4 Treatment of unbound fragments

We remind the reader that we aim to apply the R-matrix theory to processes like:

$$i \underset{\beta}{\rightarrow} A \rightarrow 1 + B \tag{47}$$

where the fragment B then subsequently decays to fragments 2 and 3. We also remind the reader that the internal energies E_λ and the reduced widths $\gamma_{\lambda,c}$ - parameters in the theory - are determined from fits to experimental data.

There is a very important issue in this decay that must be treated: The fragment B is not a stable fragment - it decays, and so it is not trivially treated in R-matrix theory. In essence, the mass of B (m_B) is not locked to something specific, but can vary appreciably if the states of B are not very narrow. This results in a continuum of final channels for the first fragmentation. This is not a theoretical problem, as shown in [8] chapter $XIII$, section 2, but it is indeed a practical one: whereas before we had a countable number of couplings γ between external and internal states, each coupling is now a continuous function of the mass of fragment B. Since these couplings are not calculated, but instead are taken from experiments, this makes the theory useless. Let us elaborate on the issue and see how it may be solved. Though the mass of the recoil B is fully determined by the final state (as shown in section 2), we have to acknowledge that the coupling $\gamma_{1B,A}$ and the phase and penetrabililty all depend on the mass of the recoil:

$$U_{i,1+B} = i\mathbf{\Omega}_{1B,m_B} \sqrt{2\mathbf{P}_{1B,m_B}} \gamma_{1B,A}(m_B) \mathbf{A} \sqrt{f_\beta} g_{A,i} \tag{48}$$

Also, we must include an integral over m_B in the definition of the level matrix A, so that we indeed sum / integrate over all outgoing channels:

$$(A^{-1})_{i,j} = \delta_{i,j} \times (E_i - E) - \sum_c \int dm_B \gamma_{i,c}(m_B)\gamma_{j,c}(m_B)(S_c - B_c + iP_c) \quad (49)$$

Here of course, the shift functions, boundary conditions and penetrabilities also depend on the mass of m_B. We want to eliminate the dependence on m_B in the expression for the level matrix. To do so, let us examine where exactly m_B enters the expression for S_c, B_c and P_c. Remember that the shift function is given in terms of the Coulomb wave functions (see equation 36):

$$S_c = k_c a_c \frac{F_c \dot{F}_c + G_c \dot{G}_c}{F_c^2 + G_c^2}$$

Here, k is the wavenumber of the channel (it depends on the kinetic energy), and F and G are the Coulomb wave functions (they also depend on the kinetic energy of the channel) evaluated at the channel radius a_c. So, the shift function is a function of the kinetic energy in the channel:

$$S_c = S_c(K_c), \quad K_c = m_A - m_1 - m_B \quad (50)$$

We want to eliminate the dependence on m_B: we can try to eliminate this dependency by integrating. Let us define a mean shift function $\bar{S}_c$:

$$\bar{S}_c(m_A) \equiv \int dm_B S_c(m_A - m_1 - m_B)\rho_c(m_B) \quad (51)$$

Note that m_1 is suppressed on the left hand side, as this is simply a constant. The symbol $\rho_c(m_B)$ is the shape of the B-state accessed in the channel c. It is included as a weight in the integral. Though it is not known[7], a first approximation might be to use a Breit-Wigner shape with experimental position and width - parameters related to the internal energies E_λ and the reduced widths $\gamma_{\lambda,c}$ that we already need in our model:

$$\rho_c(m_B) = \frac{1}{2\pi} \frac{\Gamma_c}{|E_c^0 - m_B + i\Gamma_c/2|^2} \quad (52)$$

[7]The shape of the level is one of the things we hope to describe using our model.

Inspired by this "trick", we might attempt removing the integral in the expression for the level matrix as well: We propose the following approximation for the γ's:

$$\gamma_{i,c}(m_B) \approx \bar{\gamma}_{i,c}\sqrt{\rho_c(m_B)} \tag{53}$$

The intuition for the approximation is this: The further from the resonance the outgoing channel is, the smaller the wave function - hence also a smaller overlap integral on the surface. The Breit-Wigner shape is normalized such that the integral over equation 53 gives the averaged reduced width $\bar{\gamma}_{i,c}$. Using this expression in the level matrix formula we get:

$$(A^{-1})_{i,j} = \delta_{i,j} \times (E_i - E) - \sum_c \bar{\gamma}_{i,c}\bar{\gamma}_{j,c}(\bar{S}_c - B_c + i\bar{P}_c) \tag{54}$$

where $\bar{P}_c$ is defined completely analogously to $\bar{S}_c$. The boundary condition parameter can now also be defined (without any relation to m_b) as:

$$B_c = \bar{S}_c(m_A = m_A^0) \tag{55}$$

where m_A^0 refers to the position of the lowest resonance in A. This method is inspired by the discussion in [6], but expands the intuition for the definition of the mean shift and penetrability.

We also must discuss the mass dependent couplings $\gamma_{1B,A}(m_B)$ appearing directly in the amplitude. We can not simply use equation 53 because it includes the fragmentation of the recoil fragment B (the fragmentation is what produces the width), and we will describe that part of the process separately. Instead we use the averaged reduced widths $\bar{\gamma}_{1B,A}$.

4.5 Conservation of angular momentum

We shall include a comment on the appearence of angular momentum coupling coefficients in the theory. The coupling coefficients must of course appear somewhere to ensure conservation of angular momentum. Some choose to include these couplings in the link between the amplitude and the collision-matrix (e.g. [8]), but we shall instead contain them in the reduced widths $\gamma_{c,\lambda}$ and $g_{\lambda,i}$. That is, in our formulation the reduced widths contain angular momentum coupling coefficients such as:

$$\gamma_{c,\lambda} \propto \langle S_1, m_{S_1}, S_2, m_{S_2}, l, m_l | S_\lambda, m_{S_\lambda} \rangle \tag{56}$$

where S_1 and S_2 refer to intrinsic spins of the two channel fragments, l refer to the relative orbital angular momentum in the channel, and S_λ refer to the intrinsic spin of the internal state λ. This means that even though we work in a basis of all the spin-projection substates we do not have freedom to fit independent reduced widths between all of these as many of the reduced widths are related to one another through angular momentum coupling rules. These couplings are not just Clebsch-Gordan coefficients since we couple *three* angular momenta (S_1, S_2 and l) to one. However, they can be calculated. We shall demonstrate with a more general example. We would like to calculate products of the form:

$$\langle S_1, m_1; S_2, m_2; l, m_l | S_i, m_i \rangle \tag{57}$$

We begin by coupling $\vec{S_1} + \vec{S_2} = \vec{S}$ by inserting a resolution of identity:

$$\sum_{S=|S_1-S_2|}^{S_1+S_2} \langle S_1, m_1; S_2, m_2; l, m_l | S, m_1+m_2; l, m_l \rangle \langle S, m_1+m_2; l, m_l | S_i, m_i \rangle \tag{58}$$

We have explicitly used the conservation of angular momentum projection, removing the sum over m_S. We split out the l, m_l part from the first factor, leaving a Clebsch-Gordan coefficient:

$$\sum_S \langle S_1, m_1; S_2, m_2 | S, m_1 + m_2 \rangle \langle S, m_1 + m_2; l, m_l | S_i, m_i \rangle \tag{59}$$

We recognize the second factor as another Clebsch-Gordan coefficient, coupling $\vec{S} + \vec{l} = \vec{S_i}$. Hence, we conclude $m_i = m_1 + m_2 + m_l$. In total, we define:

$$\begin{aligned} T_{1+2+l\to i}(S_1, m_1; S_2, m_2; l, m_l; S_i, m_i) &= \delta_{m_1+m_2+m_l, m_i} \\ &\times \sum_S \langle S_1, m_1; S_2, m_2 | S, m_1 + m_2 \rangle \langle S, m_1 + m_2; l, m_l | S_i, m_i \rangle \end{aligned} \tag{60}$$

This sum over products of Clebsch-Gordan coefficients is included in the reduced widths. The observant reader may have noticed that this section is closely related to Racah's W-coefficients, or Wigner's $6j$-symbols. Both the W-coefficients and the $6j$-symbols deal with angular momentum recoupling: they convert between bases with different orders of coupling [14]. We chose to couple the two intrinsic spins first, and then the orbital angular momentum,

and the work of Racah and Wigner allows to relate these coefficients to those where we couple one of the intrinsic spins to the orbital angular momentum first, and then couple the sum to the other intrinsic spin. We don't need these recoupling coefficients (the $6j$ and the W) in the development of our model.

5 Calculating the rate

Armed with an understanding of three-body quantum mechanics and R-matrix theory we shall now delve into the problem at hand: We want to calculate the rate of a process where a nucleus i β-decays to a nucleus A, which then decays to three fragments a, b and c. Schematically:

$$i \underset{\beta}{\rightarrow} A \rightarrow a + b + c \tag{61}$$

If the initial nucleus has spin S_i and the final fragments have spins S_a, S_b, S_c we may write:

$$\frac{d^3w}{d\Omega_a d\Omega_b d\Omega_c} = \frac{1}{2S_i+1} \sum_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}} |\mathcal{A}_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}}|^2 \tag{62}$$

for an unpolarized initial state, and unpolarized measurements of the final fragments, when $\mathcal{A}_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}}$ is the total amplitude of the process with a given set of spin projections[8]. We shall imagine that the final fragments are detected by momentum measurements $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c$[9], and hence the amplitude is a function of these momenta - both their magnitude and direction.

Three-body problems are very hard, and so we shall try to treat this problem as a sequence of two two-body processes instead. Specifically, we imagine that one fragment (either a, b or c) breaks off first leaving a recoil B (that is not measured), which then subsequently breaks apart into the remaining fragments. Schematically:

$$i \underset{\beta}{\rightarrow} A \rightarrow 1 + B \rightarrow 1 + 2 + 3. \tag{63}$$

[8]The observant reader may notice that the expression on the right has no unit, while the expression on the left clearly do. This is related to the normalization constant mentioned in section 4.3.

[9]There is no such thing as a "momentum detector" - in a real experiment *the energy* would be measured in a specific point. Then, the momentum is *reconstructed* from the energy, the point of measurement and the (known) point of the source.

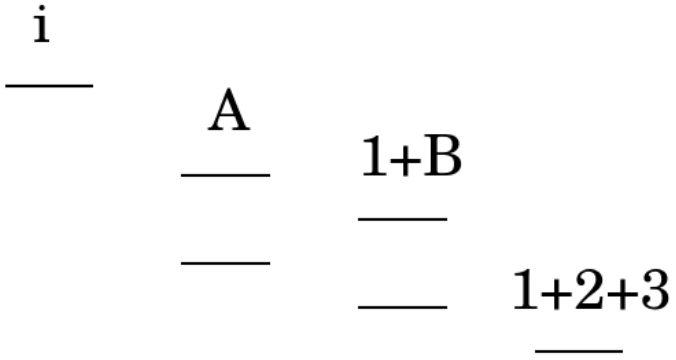

Figure 5: A diagram of the sequential process. A state of the nucleus i β-decays to a set of states of the nucleus A. Nucleus A breaks up into two fragments 1 and B, and then at last B breaks up into the remaining two fragments 2 and 3. The total process is modelled as *a sequence of two-body fragmentations*.

It is important that the second fragmentation happens after the first fragment is sufficiently far away for it to not affect the second fragmentation in any way. Otherwise the process would not be sequential and a two-body treatment of the second decay would not be justified.

This sequential model of the total process is illustrated in figure 5. Note that the recoil nucleus B may change depending on which fragment is emitted first. Since we do not measure which fragment breaks off first we must sum over all processes. That is, we decompose the final state into three parts. Each part is described in the appropriate set of Jacobi coordinates as described in section 3 (see figure 6), e.g. the sequence where a is emitted first is described in the coordinates where the subsystem consists of b and c and so forth. We remind the reader that the canonical momenta of the Jacobi coordinates are given by:

$$\mathbf{p}_1 = \mu_{a,B}(\mathbf{p}_a/m_a - \mathbf{p}_B/m_B), \quad \mathbf{p}_{23} = \mu_{b,c}(\mathbf{p}_b/m_b - \mathbf{p}_c/m_c), \qquad \mu_{b,c} = \frac{m_b m_c}{m_b + m_c} \tag{64}$$

where the notation has been slightly changed compared to section 3 to emphasize that $\mathbf{p}_1$ describes the first fragment. We mentioned earlier that any permutation of the three fragments could be used (we may switch a,b and c

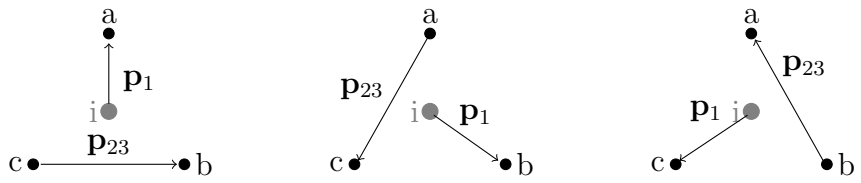

Figure 6: The three different sets of Jacobi coordinates, corresponding nicely to the three different sequences of break-ups. Note that the scaling of the momentum vectors is not indicated in this drawing, only the direction.

in equation 64), but we shall only include three here: $(a, b, c), (b, c, a), (c, a, b)$ - i.e. all cyclic permutations. We do so following [9]. Note that the acyclic permutations are almost identical to the cyclic ones - they only differ in the direction of the relative coordinate of the two fragments in the quasi-bound system, and so describe the same physics, but in a slightly different way[10]. We also remind the reader that here B is the recoil, i.e. the quasi-bound state of fragments $2 + 3$, and m_B refers to the rest energy of the recoil. The reduced mass of a and the recoil B is not written out explicitly, but is defined in analogy with $\mu_{b,c}$.

Armed with this description we shall write out the total amplitude (for given projections of initial and final states) as a sum over the three sequential contributions, which we call the *sequential amplitudes* $\mathcal{A}_1$:

$$\mathcal{A}_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}} = \sum_{1=a,b,c} \langle \mathbf{p}_1, \mathbf{p}_{23}, m_{S_1}, m_{S_2}, m_{S_3} | m_{S_i} \rangle \equiv \sum_{1=a,b,c} \mathcal{A}_1 \qquad (65)$$

If we suppress the initial and final state spin projections in our notation (and use shorthand notation $d\Omega = \Pi_j d\Omega_j$), we may express the differential rate as

[10]The same issue arises in the two-body problem: consider for example the hydrogen atom, where an electron and a proton orbit. Should we choose the relative coordinate as going from the proton to the electron or vice versa? We can of course do either, but one is sufficient. Since we include three *different* contributions in the three-body version we must take care to be internally consistent (the amplitudes may depend on the direction of the relative coordinate of the subsystem), and we can not choose *any* three sets of coordinates. It turns out that whichever one we choose first, the following two must be chosen as the cyclic permutations. See [9], bottom of page 380.

a function of the sequential amplitudes:

$$\frac{dw}{d\Omega} = \frac{1}{2S_i+1} \sum_{m_{S_i},m_{S_a},m_{S_b},m_{S_c}} |\mathcal{A}|^2 = \frac{1}{2S_i+1} \sum_{m_{S_i},m_{S_a},m_{S_b},m_{S_c}} \left| \sum_{1=a,b,c} \mathcal{A}_1 \right|^2 \tag{66}$$

Our task is now to calculate the sequential amplitude, $\mathcal{A}_1$.

5.1 Sequential Amplitude

We want to calculate the sequential amplitude of going from a state i, via our sequential process (represented by an operator T), to a three-particle final state of definite linear momentum (a plane wave state):

$$\mathcal{A}_1 = \langle \mathbf{p}_1, \mathbf{p}_{23} | T | i \rangle. \tag{67}$$

Here the spin projections of the initial state and final fragments are implicit, to not overload the notation. We know the process goes through a β-decay to states in a nucleus A; hence we shall represent the operator T as set of states of this nucleus, in analogy with equation 32. Let us characterize these states by their spin-parity S^π_A and their spin projection m_{S_A}, and name the collection of these two quantum numbers $\lambda = (S^\pi_A, m_{S_A})$. We write:

$$\mathcal{A}_1 = \sum_\lambda \langle \mathbf{p}_1, \mathbf{p}_{23} | \lambda \rangle \langle \lambda | i \rangle \tag{68}$$

Since we are going to model this process sequentially, we now introduce another set of states: namely a set of states of the first fragment 1, and the recoil nucleus B, that we don't measure directly in the experiment. These states contain relative degrees of freedom: orbital angular momentum l and its projection m_l; as well as internal degrees of freedom of the recoil nucleus B: S^π_B, m_{S_B}. Collectively we shall name these four quantum numbers as $\mu = (l, m_l, S^\pi_B, m_{S_B})$. We write:

$$\mathcal{A}_1 = \sum_{\lambda,\mu} \langle \mathbf{p}_1, \mathbf{p}_{23} | \mu \rangle \langle \mu | \lambda \rangle \langle \lambda | i \rangle \tag{69}$$

Now we examine the matrix element $\langle \mathbf{p}_1, \mathbf{p}_{23} | \mu \rangle$ more closely:

$$\langle \mathbf{p}_1, \mathbf{p}_{23} | \mu \rangle = \langle \mathbf{p}_1, \mathbf{p}_{23} | l, m_l, S^\pi_B, m_{S_B} \rangle \tag{70}$$

Assuming the final state fragments do not interact we may split the final state into a product state: $\langle \mathbf{p}_1, \mathbf{p}_{23}| = \langle \mathbf{p}_1|\langle \mathbf{p}_{23}|$. Now, the product in equation 70 represents the decay of the recoil fragment B into the fragments $2+3$, and in our sequential model we assume this process to happen only when the first fragment is far away; hence we may also split the state $|l, m_l, S_B^\pi, m_{S_B}\rangle$ appearing here into a product state: $|l, m_l, S_B^\pi, m_{S_B}\rangle = |l, m_l\rangle|S_B^\pi, m_{S_B}\rangle$. We may now write:

$$\langle \mathbf{p}_1, \mathbf{p}_{23}|\mu\rangle = \langle \mathbf{p}_1|l, m_l\rangle\langle \mathbf{p}_{23}|S_B^\pi, m_{S_B}\rangle \tag{71}$$

Now, the first matrix element is often seen in scattering theory; see e.g. [11] p. 392. It can be calculated using a spherical harmonic:

$$\langle \mathbf{p}_1|l, m_l\rangle = \frac{\hbar}{\sqrt{\mu_{1,B}k_1}} Y_l^{m_l}(\hat{\mathbf{p}}_1), \quad \hbar k_1 = |\mathbf{p}_1| \tag{72}$$

The other matrix element demands a little more attention: We insert a basis of spherical waves $|l_2, m_{l_2}\rangle$ in the relative coordinates of the fragments 2 and 3:

$$\langle \mathbf{p}_{23}|S_B^\pi, m_{S_B}\rangle = \sum_{l_2, m_{l_2}} \langle \mathbf{p}_{23}|l_2, m_{l_2}\rangle\langle l_2, m_{l_2}|S_B^\pi, m_{S_B}\rangle \tag{73}$$

and note that the first matrix element (from the left) appearing here is familiar: it is also a spherical harmonic. The second matrix element represents the coupling from a state of the recoil nucleus B to a state of $2+3$ with definite angular momentum. We shall treat this coupling in the center-of-mass system of the recoil nucleus, and hence the orbital angular momentum will also be evaluated in this system. For completeness, the momentum $\mathbf{p}_{23}$ should also be transformed to this system via a Lorentz transformation (since it is used in calculations in this system). However, unless we are dealing with relativistic energies the relative momentum $\mathbf{p}_{23}$ is in a sense already in the recoil center of mass system.

Instead of equation 70 we now have:

$$\langle \mathbf{p}_1, \mathbf{p}_{23}|\mu\rangle = \sum_{l_2, m_{l_2}} \frac{\hbar}{\sqrt{\mu_{1,B}k_1}} Y_l^{m_l}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3}k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23})\langle l_2, m_{l_2}|S_B^\pi, m_{S_B}\rangle \tag{74}$$

Now, we propose something that might seem slightly odd. We would like to perform the same calculation again, but slightly different. Let us insert a

larger basis $|l', m_{l'}, l_2, m_{l_2}\rangle$ where l' refers to the orbital angular momentum of 1 and B, and l_2 still refers to the orbital angular momentum of 2 and 3 instead of the one we just used:

$$\langle \mathbf{p}_1, \mathbf{p}_{23}|\mu\rangle = \sum_{l',m_{l'},l_2,'m_{l_2}} \langle \mathbf{p}_1, \mathbf{p}_{23}|l', m_{l'}, l_2,' m_{l_2}\rangle\langle l', m_{l'}, l_2,' m_{l_2}|\mu\rangle \tag{75}$$

The difference between this step and the one leading to equation 73 is that here we insert a set of three-particle states, while the states in 73 are only two-particle states. We know that only terms with $l' = l$ and $m_{l'} = m_l$ will contribute to the sum, i.e. the coupling from B to $2 + 3$ should not alter the state of 1 - otherwise it wouldn't be a sequential process. We repeat the above arguments and arrive now at:

$$\langle \mathbf{p}_1, \mathbf{p}_{23}|\mu\rangle = \sum_{l',m_{l'},l_2,m_{l_2}} \frac{\hbar}{\sqrt{\mu_{1,B}k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3}k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \times \langle l', m_{l'}, l_2, m_{l_2}|\mu\rangle \tag{76}$$

Now, if we define $c = (l', m_{l'}, l_2, m_{l_2})$ we may write:

$$\langle \mathbf{p}_1, \mathbf{p}_{23}|\mu\rangle = \sum_c \frac{\hbar}{\sqrt{\mu_{1,B}k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3}k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \langle c|\mu\rangle \tag{77}$$

Now, let us return to the sequential amplitude expression, and substitute our result:

$$\mathcal{A}_1 = \sum_c \frac{\hbar}{\sqrt{\mu_{1,B}k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3}k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \sum_{\lambda,\mu} \langle c|\mu\rangle\langle\mu|\lambda\rangle\langle\lambda|i\rangle \tag{78}$$

We have now coupled the outgoing spherical waves from the break-up process (the channels c, with definite angular momentum in both relative coordinates) to plane waves of definite linear momenta in the two relative coordinates (the $\mathbf{p}_1$,$\mathbf{p}_{23}$ states). The double sum over λ, μ describing the coupling from the initial state to the spherical waves must now be calculated.

A word on the angular dependence is in place here. We mentioned in section 2 that the amplitude must be independent of the direction of the first emitted fragment if the initial state is isotropic. The observant reader may have noticed that the sequential amplitude is *explicitly* dependent on the direction of the first emitted fragment. This is because the sequential amplitude we have just calculated assumes a *specific spin projection of the initial state*, and the directional dependence doesn't vanish until the sum over initial state projections is carried, as we did in equation 66.

5.2 Three-body R-matrix amplitude

Now, why did we go through this strange extra step in the end? We did so because the last sum has a specific form:

$$\mathcal{A}_{c,i} = \sum_{\lambda,\mu} \langle c|\mu\rangle\langle\mu|\lambda\rangle\langle\lambda|i\rangle \tag{79}$$

where c is a channel of definite angular momenta - unlike the expression in equation 69 with a channel of definite linear momenta. This amplitude looks much like the amplitudes treated in R-matrix theory. Specifically, we remind the reader of equation 32:

$$\mathcal{A}_{c',c} = \sum_{\lambda} \langle c'|\lambda\rangle\langle\lambda|c\rangle$$

We note that equation 79 looks like a combination of *two R-matrix amplitudes* like the one in equation 32, with the outgoing channels of the first process becoming the internal levels of the second process. Schematically we might represent this idea as follows:

$$i \rightarrow \boxed{A \rightarrow 1 + B} \boxed{\rightarrow 1 + 2 + 3}$$

The upper left (red) box is the first R-matrix process, a β-decay through states of A resulting in fragments $1 + B$, while the lower right (blue) box is the second R-matrix process, a state A breaking apart sequentially first by emitting fragment 1 leaving a recoil B, which then subsequently breaks apart into $2 + 3$. As indicated by the boxes the two processes overlap: The internal levels of the second process (the states in B) are fed by the decay of the internal levels of the first process (the states in A). We must be careful to not include this overlap twice.

Let us follow this idea and calculate the amplitude in equation 79 by combining two R-matrix expressions. First we write out the amplitude for the first decay:

$$\mathcal{A}_{\mu,i} = \sum_{\lambda} \langle\mu|\lambda\rangle\langle\lambda|i\rangle \tag{80}$$

where a channel μ specifies the quantum numbers $\mu = S_1, m_{S_1}, S_2, m_{S_2}, l, m_l$ and where the sum over λ is a sum over all states of the internal system. Including the corrections for β-decay channels, and the corrections needed

to compensate for unbound final state fragments we may write an expression for the collision matrix of going from initial state i to a state μ, as defined in section 4:

$$\vec{U}_{\mu,i} = \mathbf{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f} \tag{81}$$

We remind the reader that the states i are characterized by quantum numbers $i = S_i^\pi, m_{S_i}$, the states λ are characterized by quantum numbers $\lambda = S_A^\pi, m_{S_A}$ and the states μ are characterized by quantum numbers $\mu = S_1^\pi, m_{S_1}, S_B^\pi, m_{S_B}, l, m_l$. Also, the matrices $\mathbf{\Omega}_\mu$ and $\sqrt{\mathbf{2P}}_\mu$ are diagonal matrices. Also, the sum over states appearing in the level matrix $\mathbf{A}$ must of course include *all* accessible states from A, i.e. states of all the possible recoil nuclei (the different B's), not just the ones available in the sequence we are currently calculating[11].

For the full process $i \to c$ we must generalize the collision matrix. The states μ changes from being *channels* as they were in the first part of the process, to being internal levels in the second part of the process. We have already coupled into these states - what remains is the broadening of these states, and the coupling out of them. So, to avoid redundancy we must simply include a level matrix $\mathbf{B}$, and a coupling from the internal states μ to the final channels c. We propose the following expression based on this idea:

$$U_{c,i} = \Omega_c \sqrt{2P_c} \gamma_{c,\mu} \mathbf{B} \mathbf{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f} \tag{82}$$

where of course the couplings $\gamma_{c,\mu}$ vanish unless $l = l'$ and $m_l = m_{l'}$ - i.e. we only couple states μ to final channels c if the first fragment remains in the same state. The above expression is in essence two R-matrix processes in succession - with the added complication that the final channels of the first process become the internal levels of the second process.

Returning to the amplitude in equation 79, we remember that this amplitude is (essentially) equal to the collision matrix we just calculated up to a constant of proportionality which must be fitted to the half-life of the β-decay anyway. Hence, we shall simply insert equation 82 into equation 78 and write out the expression for the sequential amplitude $\mathcal{A}_1$ as:

[11] This is because the sum over states in the definition of the level matrix is essentially what causes the broadening (because it couples the internal states to the continuum), and the widths of the internal levels depend on *all* accessible continuum states.

$$\mathcal{A}_1 = \sum_c \frac{\hbar}{\sqrt{\mu_{1,B} k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3} k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \\ \times \Omega_c \sqrt{2P_c} \gamma_{c,\mu} \mathbf{B}\mathbf{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f} \quad (83)$$

This is the main result of this thesis.

6 Example

To make our proposed sequential model more concrete we shall include an example that may be treated by this model. Hopefully, this example also indicates how fast the size and complexity of the model grows.

As the triple-α break up of ^{12}C reduces some of the complexity since all fragments are spin-0, we shall use this as an example. Note that this is also the subject of the investigations by Balamuth et al. [5] mentioned in the introduction. Contrary to Balamuth et al. we imagine populating states in ^{12}C by a β^--decay from ^{12}B. The triple-α threshold is 7.276 MeV above the ground state of ^{12}C, meaning we will only consider states above this threshold energy. Three ^{12}C states above this threshold are populated by the β-decay, meaning we consider the states $0^+, 0^+, 1^+$ at 7.654 MeV (the so-called Hoyle state), 10.3 MeV and at 12.71 MeV (one of the states considered by Balamuth) respectively. These states all decay through α-emission to ^{8}Be, followed by a fragmentation into two α-particles. We shall include two states in ^{8}Be: the 0^+ ground state, and the first excited 2^+ state at 3.03 MeV. Since all three fragments are identical, all sequences go through the same recoil (^{8}Be). This also simplifies the model. The allowed values of angular momentum of both the first fragment (l) and the last two fragments (l_2) can be calculated to be either 0 (s-wave) or 2 (d-wave). See figure 7.

The rate is then computed as a sum over the initial and final state projections:

$$\frac{d^3 w}{d\Omega_a d\Omega_b d\Omega_c} = \frac{1}{2S_i + 1} \sum_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}} |\mathcal{A}_{m_{S_i}, m_{S_a}, m_{S_b}, m_{S_c}}|^2 \quad (84)$$

However, since all final state fragments are spin 0 the sums over their pro-

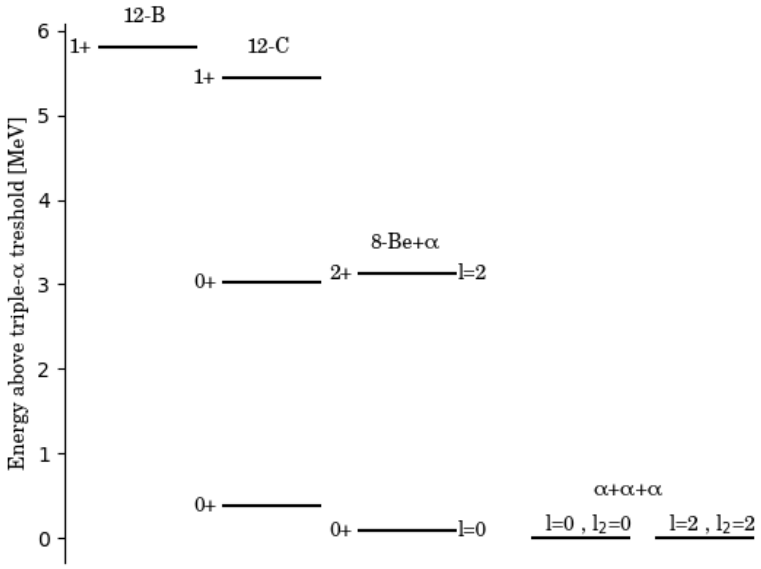

Figure 7: A level diagram of the relevant levels in the β-decay of ^{12}B to ^{12}C, followed by a sequential triple-α decay through ^{8}Be. l refers to the angular momentum of the first α relative to the recoil nucleus (^{8}Be), while l_2 refers to the relative angular momentum between the two last α's. The spin-projection substates have not been shown on the diagram, to avoid confusion.

jections drop out.

$$\frac{d^3w}{d\Omega_a d\Omega_b d\Omega_c} = \frac{1}{2\cdot 1+1}\sum_{m_{S_i}=-1}^{1} |\mathcal{A}_{m_{S_i}}|^2 \tag{85}$$

The example has 5 λ-levels in ^{12}C: one level for each 0^+ state, and 3 levels for the 1^+ state. Note that only 3 reduced widths are indeed independent in $g_{\lambda,i}$, since the couplings to the different substates in 1^+ are related by angular momentum coupling. The number of μ-levels in this example is 26: one level from the $0^+, l = 0$ state and 5×5 levels from the $2^+, l = 2$ state corresponding to all combinations of projections of the spin of ^{8}Be and l. Again, we do not need 5×26 independent reduced widths to describe the coupling $\gamma_{\mu,\lambda}$ - we only need 6 (3 λ-states times 2 μ-states). Finally, we have $6 = 1 + 5$ final channels available for the $^8\text{Be} \to \alpha + \alpha$ decay (1 from $l = 0$ and 5 from $l = 2$). Four reduced widths are necessary here (and two of them are even zero, since the 0^+-state can only couple to $l_2 = 0$ and so on!).

Even though this is one of the simplest use-cases for this model, the calculation of the sequential amplitude is already quite involved. Now, had we instead considered the β-decay of ^{9}Li to ^{9}Be followed by a break-up to $\alpha+\alpha+n$ the complexity of a fragment with non-zero spin would be added, *and* the different sequential amplitudes would go through different recoil nuclei (when the *neutron* breaks off first the reaction goes through ^{8}Be as in this example, whereas when an α breaks off first the reaction goes through levels in ^{5}He) overall resulting in a rather more complicated calculation. In all cases, one should not be fooled by the rather simple formulas - the notation hides a lot of the complications that appear when one actually tries to implement the model.

7 Relation to the three-body model of Lane & Thomas

We shall show that our sequential amplitude expression reproduces a notable result from the literature on sequential three-body decays: namely one from the aforementioned R-matrix review by Lane & Thomas [8]. Their treatment of sequential three-body decays is found in chapter XIII. It is in many ways quite similar to the model proposed in this thesis, but their treatment is

not quite as elaborate, and focuses on the simple case of two single level processes.

Lane and Thomas start by including unbound fragments in their channel space, analogous to what we did in section 4.4. Then they calculate the amplitude for going from a "normal" channel c to a channel containing three particles, where two of them came from an unbound fragment. This channel is no longer sufficiently described by the same quantum numbers used for the regular (two-body) channels - instead they further specify the energy of the unbound fragment, as well as a label of the open channel of the unbound fragment that produced the last two fragments. Let us call the unbound fragment B to match our notation. It's energy will then be E_B. Let us also name the open channels of B r following the notation in [8]. A channel c' with an unbound fragment decaying to a channel r is then called (c', E_B, r) - this is the channel containing three particles, and it corresponds to the final channels c in our notation. Lane and Thomas then proceed to describe the process by a single R-matrix expression, including the corrections from an unbound fragment. This gives an expression for the amplitude analogous to our equation 48 in section 4.4, without the β-decay correction, and with only one final state:

$$\mathcal{A}_{c,(c',E_B,r)} = \Omega_{(c',E_B,r)}\sqrt{2P_{(c',E_B,r)}}\gamma_{\lambda,(c',E_B,r)}\mathbf{A}\Omega_c\sqrt{2P_c}\gamma_{c,\lambda}\gamma_{1B,A}(m_B) \quad (86)$$

Lane and Thomas only consider a single level λ in the process producing the unbound fragment, and hence the level matrix reduces to a *single number*. That makes the inversion trivial, and the level "matrix" is given by:

$$\mathbf{A} = \frac{1}{E_\lambda - E - \sum_{c'',r'}\int dE'_B\gamma^2_{\lambda,(c'',E'_B,r')}\left[S_{c''} - B_{c''} + iP_{c''}\right]} \quad (87)$$

where the sum over channels appearing in the denominator has been generalized to a continuum of channels, like we did in section 4.4. This expression is then used in the amplitude:

$$\mathcal{A}_{c,(c',E_B,r)} = \frac{\Omega_c\sqrt{2P_c}\gamma_{c,\lambda}\Omega_{(c',E_B,r)}\sqrt{2P_{(c',E_B,r)}}\gamma_{\lambda,(c',E_B,r)}}{E_\lambda - E - \sum_{c'',r'}\int dE'_B\gamma^2_{\lambda,(c'',E'_B,r')}\left[S_{c''} - B_{c''} + iP_{c''}\right]} \quad (88)$$

At this point Lane and Thomas introduce a bit of notation:

$$\Gamma_{c,\lambda} = 2P_c\gamma^2_{c,\lambda} \tag{89}$$

$$\Delta_\lambda = -\sum_{c,r}\int \gamma^2_{\lambda,(c,E_B,r)}[S_c - B_c]dE_B \tag{90}$$

$$\Gamma_\lambda = \sum_{c,r}\int 2P_c\gamma^2_{\lambda,(c,E_B,r)}dE_B \tag{91}$$

which makes the amplitude expression look like:

$$\mathcal{A}_{c,(c',E_B,r)} = \frac{\Omega_c\sqrt{\Gamma_{c,\lambda}}\Omega_{(c',E_B,r)}\sqrt{\Gamma_{\lambda,(c',E_B,r)}}}{E_\lambda + \Delta_\lambda - E - \frac{i}{2}\Gamma_\lambda} \tag{92}$$

and makes the norm-square of the amplitude:

$$|\mathcal{A}_{c,(c',E_B,r)}|^2 = \frac{\Gamma_{c,\lambda}\Gamma_{\lambda,(c',E_B,r)}}{(E_\lambda + \Delta_\lambda - E)^2 + \Gamma^2_\lambda/4} \tag{93}$$

Lane and Thomas then proceed to point out that the coupling $\gamma_{\lambda,(c',E_B,r)}$ is an unknown function of E_B, like we mentioned in section 4.4. To solve this problem, they consider a single intermediate level μ in B "between" the level λ in A and the final channel (c', E_B, r). In their model, the second decay is hence included as an energy dependence in the reduced width $\gamma_{\lambda,(c',E_B,r)}$. Schematically:

$$c \to \lambda \to (c', \mu) \text{ with energy } E_B \to (c', E_B, r). \tag{94}$$

Then they describe the decay through the level μ with a single level expression:

$$\gamma^2_{\lambda,(c',E_B,r)} \approx \frac{1}{2\pi}\gamma^2_{\lambda,(c',\mu)}\frac{\Gamma_{\mu,r}}{(E_\mu + \Delta_\mu - m_B)^2 + \Gamma^2_\mu/4} \tag{95}$$

where m_B refers to the rest energy of the unbound fragment B. Note the factor of $1/2\pi$ included in the definition of the reduced width. This is to ensure that our interpretation of the reduced width is intact. To see this, assume that the level μ is relatively narrow, compared to the variations of the shift and penetrability. Then we may integrate over the energy of the unbound fragment and sum out all open channels of the unbound fragment r:

$$\sum_r \int \gamma^2_{\lambda,(c',E_B,r)}dE_B = \gamma^2_{\lambda,(c',\mu)} \tag{96}$$

Had we not included the $1/2\pi$ they would have appeared here (from the integral of the Breit-Wigner shape), meaning the total width would no longer be given by $\Gamma = 2P\gamma^2$. Hence, the $1/2\pi$ is a normalization of the second level. In total the norm square of the amplitude is then:

$$|\mathcal{A}_{c,(c',E_B,r)}|^2 = \frac{\Gamma_{c,\lambda}\Gamma_{\lambda,(c',\mu)}}{(E_\lambda + \Delta_\lambda - E)^2 + \Gamma_\lambda^2/4}\frac{1}{2\pi}\frac{\Gamma_{\mu,r}}{(E_\mu + \Delta_\mu - m_B)^2 + \Gamma_\mu^2/4} \quad (97)$$

This is completely analogous to our expression in equation 82, except the level matrices $\mathbf{A}$ and $\mathbf{B}$ have been given explicitly for the case of one level (and some notation has been introduced). We have a coupling from the initial state to the first internal state ($\Gamma_{c,\lambda}$), then a coupling from the first internal state to the internal channel containing the unbound fragment ($\Gamma_{\lambda,(c',\mu)}$), and lastly the coupling from the unbound fragment to its decay products ($\Gamma_{\mu,r}$). Lane and Thomas reaches the same result as we did, but instead of explicitly combining two R-matrix expressions they produced the expression for the second process from one of the couplings in the expression for the first process.

Since the article by Lane and Thomas is from 1958, they were limited by computational power, and so they proceed to integrate this expression over the energy of the unbound fragment, showing that the sequential three-body break-up may in fact be modelled by a simple one-level expression, provided that the decay of the unbound fragment may be described by a single-level formula, and that this single level is sufficiently narrow for the integration to be simple:

$$|\mathcal{A}_{c,(c',E_B,r)}|^2 \approx \frac{\Gamma_{c,\lambda}\Gamma_{\lambda,(c',\mu)}}{(E_\lambda + \Delta_\lambda - E)^2 + \Gamma_\lambda^2/4} \quad (98)$$

In essence, Lane and Thomas show that the three-body fragmentation through isolated levels is described by a single fragmentation with one level if the following fragmentation is narrow. Our sequential amplitude expression is very much a natural generalization of this approach, where instead of isolated levels we have the full level matrix.

8 Identical particle symmetrization

Since many of the interesting use cases for this model involve identical particles (e.g. ^{12}C and ^{9}Be), we must investigate how our amplitude behaves under the interchange of final fragments. Lane and Thomas do not investigate this

in their sequential decay analysis, while Balamuth et al. do mention symmetrization. But, the analysis of Balamuth et al. is not very precise in their distinction between different sequences of emission and exchange symmetry of identical particles. Hence, a more detailed analysis of identical particle symmetrization in sequential decays is much needed.

To not confuse the labels of the particles (a, b, c) with the measured momenta (until now $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c$), we shall name the detectors α, β, γ, and henceforth name the momenta after the detector in which they were registered: $\mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma$. This puts emphasis on the fact that the final state $\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma|$ also carries information in the order in which the momenta are listed; the first momentum is the momentum of particle a, the second that of particle b and the third that of particle c. When the particles are not identical this distinction is not necessary, since we could simply name the detectors unambiguously after the particle they detected. But, when the particles are identical we must keep track of both detector and particle labels.

8.1 Two identical spin-0 bosons

Let us begin with a simple case, where a and b are identical spin-0 bosons. In this case the spin-part of the final state is trivially symmetric under exchange, and we can focus on the spatial part. We first write out the amplitude for measuring the final state $\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma|$ (remember that the two first entries correspond to particles a and b) - we shall denote this amplitude by $\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma|i\rangle$, suppressing the operator (earlier T) that couple them:

$$\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma|i\rangle = \langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{\beta\gamma}|i\rangle + \langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\gamma\alpha}|i\rangle + \langle \mathbf{p}_{1=\gamma}, \mathbf{p}_{\alpha\beta}|i\rangle \tag{99}$$

This corresponds to the three Jacobi coordinates shown in figure 6. We also refer to figure 8, pane 1-3, where each sequential contribution has been drawn with both particle and detector labels shown. Now we interchange particles a and b:

$$\langle \mathbf{p}_\beta, \mathbf{p}_\alpha, \mathbf{p}_\gamma|i\rangle = \langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\alpha\gamma}|i\rangle + \langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{\gamma\beta}|i\rangle + \langle \mathbf{p}_{1=\gamma}, \mathbf{p}_{\beta\alpha}|i\rangle \tag{100}$$

The sequential contributions from this amplitude is shown in figure 8, pane 4-6, where we note that only the particle labels have changed: the momentum in detector α is still $\mathbf{p}_\alpha$, but it is now (in our state) associated with particle b. We have also drawn the canonical momenta of the corresponding Jacobi coordinates in figure 9. If we reorder the sequential amplitudes in

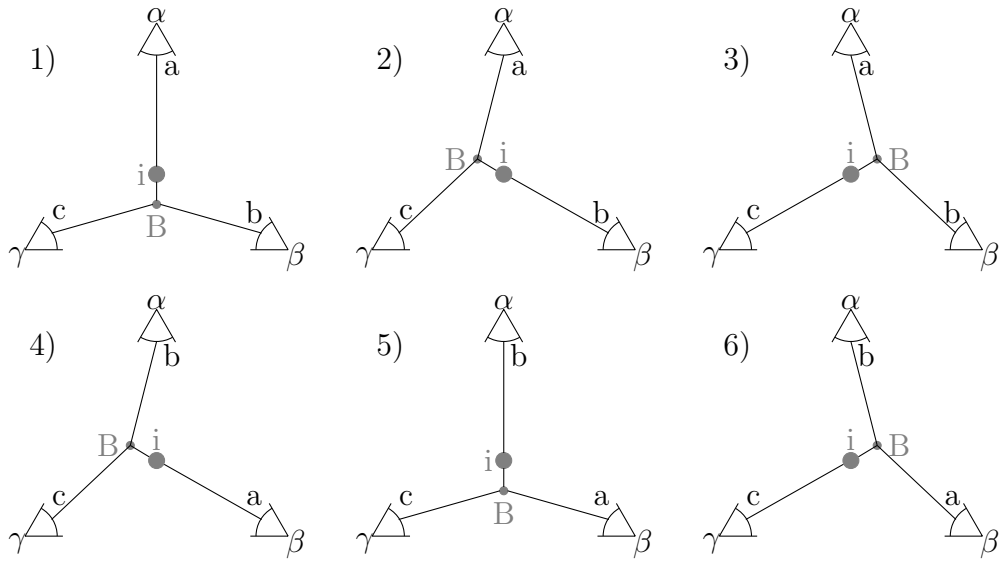

Figure 8: All six possible decays resulting in the same measurement. In the top row (1-3) we see the three decay paths where (we imagine) particle a is detected in detector α and particle b is detected in detector β. In the bottom row (4-6) we see the same three decay paths but a and b have been interchanged. Since a and b are identical and we can not tell them apart we must add all six contributions to our final amplitude.

equation 100 we see that the three amplitudes are *almost* equal to the ones appearing in equation 99 (just above it). E.g. pane 1 is very similar to pane 5, and so forth. The only difference is in the sign of the relative momentum vectors (e.g. $\mathbf{p}_{\alpha\gamma} = -\mathbf{p}_{\gamma_\alpha}$), because this depends on the order in which the momenta are listed in our states. Compare figure 9 to figure 6. We note that the interchange of two fragments corresponds to using Jacobi coordinates of all the acyclic permutations, instead of the cyclic ones.

To progress further we examine one of these sequential amplitudes. Let us choose $\langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\alpha\gamma}|i\rangle$ from equation 100 where a is emitted first (this corresponds to pane 4 in figure 8). This sequential amplitude can be calculated using equation 83. To avoid confusion we shall label the final channels (formerly c) by the letter t:

$$\langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\alpha\gamma}|i\rangle = \sum_t \frac{\hbar}{\sqrt{\mu_{a,B}k_{1=\beta}}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_{1=\beta}) \frac{\hbar}{\sqrt{\mu_{b,c}k_{\alpha\gamma}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{\alpha\gamma}) \times \Omega_t \sqrt{2P_t}\gamma_{t,\mu}\mathbf{B}\boldsymbol{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda}\mathbf{A}\vec{g}_{\lambda,i}\sqrt{f} \quad (101)$$

Let us now compare to $\langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\gamma\alpha}|i\rangle$ from equation 99, where b is emitted first:

$$\langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\gamma\alpha}|i\rangle = \sum_t \frac{\hbar}{\sqrt{\mu_{b,B}k_{1=\beta}}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_{1=\beta}) \frac{\hbar}{\sqrt{\mu_{c,a}k_{\gamma\alpha}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{\gamma\alpha}) \times \Omega_t \sqrt{2P_t}\gamma_{t,\mu}\mathbf{B}\boldsymbol{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda}\mathbf{A}\vec{g}_{\lambda,i}\sqrt{f} \quad (102)$$

Since the fragments a and b are identical, their masses, and the entire "R-matrix" part of the expressions, are too, and so the only place where these two expressions differ is in the *direction* of $\mathbf{p}_{\alpha\gamma} = -\mathbf{p}_{\gamma\alpha}$. This direction appears in the spherical harmonic $Y_{l_2}^{m_{l_2}}$. As is well known, the parity of the spherical harmonics is $(-1)^l$, so the contribution for all odd values of l_2 in the c-sum picks up a sign relative to the corresponding amplitude in equation 99. This means that each term in the t-sum in equation 102 picks up a sign $(-1)^{l_2}$ when changing the order of the last emitted fragments, e.g. $\mathbf{p}_{\alpha\gamma} \to \mathbf{p}_{\gamma\alpha}$. However, unless all terms in the sum have *either* even or odd values of l_2, this doesn't result in a symmetry of the entire sequential amplitude under this change. The values of l_2 are constrained by the conservation of parity.

The conservation of parity between the initial state P_i and the final state P_f may be expressed as:

$$P_i = (-1)^l(-1)^{l_2}P_aP_bP_c \quad (103)$$

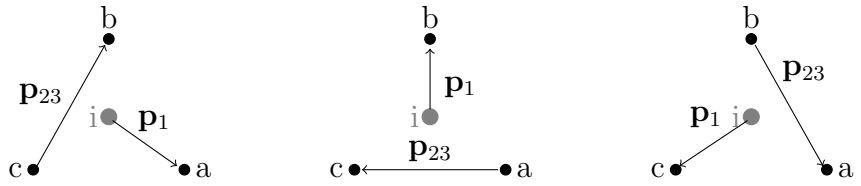

Figure 9: The three different sets of Jacobi coordinates of the interchanged system $a \leftrightarrow b$. Note that the relative momenta point the opposite way of what they do in figure 6.

where P_a is the parity of fragment a and so forth. Assuming that all final state fragments have positive parity we have:

$$P_i = (-1)^l (-1)^{l_2} \tag{104}$$

restricting the allowed values of the orbital angular momenta. Furthermore, parity must also be conserved in the first decay:

$$P_i = (-1)^l P_B \tag{105}$$

In the case that parity conservation restricts the values of l_2 to be either even or odd for all channels c in a sequential amplitude in equation 99 they have symmetry under exchange, for example:

$$\langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\alpha\gamma} | i \rangle = \pm \langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\gamma\alpha} | i \rangle \tag{106}$$

where the "+" is for even l_2-values and the "−" is for odd l_2-values. When both odd and even l_2-values contribute to the sequential amplitudes they are not symmetric under exchange of two particles. Generally, one must use a symmetric final state to calculate the amplitudes when identical particles are involved:

$$\mathcal{A} = \frac{1}{\sqrt{2}} \left(\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma | i \rangle + \langle \mathbf{p}_\beta, \mathbf{p}_\alpha, \mathbf{p}_\gamma | i \rangle \right) \tag{107}$$

8.2 The case of ^{9}Li

The β-decay of ^{9}Li to ^{9}Be followed by a break-up to $\alpha + \alpha + n$ is a case of exactly the type described above: the final state has two identical spin-0 bosons (the two α-particles). The sequential model has two different decay

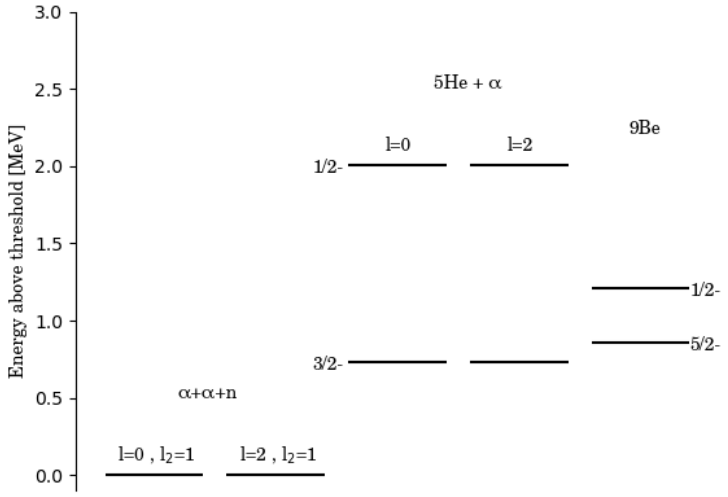

Figure 10: A diagram of levels in ^{9}Be and ^{5}He involved in the $\alpha + \alpha + n$ break-up following the β-decay of ^{9}Li. The level in ^{9}Li is not shown, and the process "runs from right to left" here.

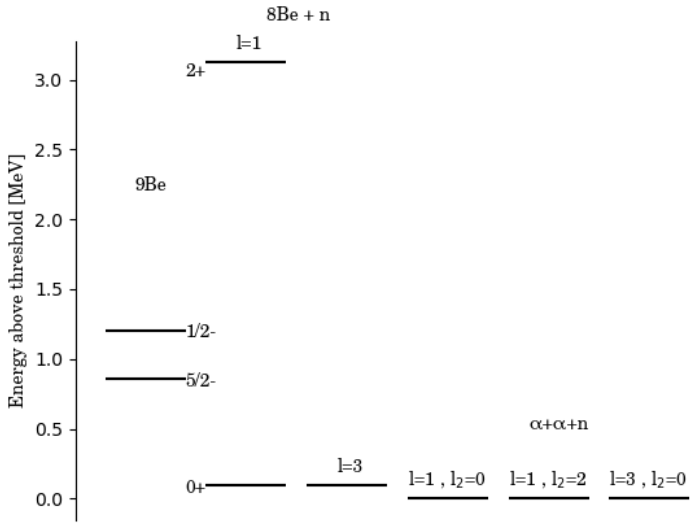

Figure 11: A diagram of levels in ^{9}Be and ^{8}Be involved in the $\alpha + \alpha + n$ break-up following the β-decay of ^{9}Li. The process "runs from left to right".

paths: one through ^{5}He and one through ^{8}Be. See figures 10 and 11 for an example of what levels to include.

Let us name the first α a, the second α b and the neutron c, and let us index the measured momenta as $\mathbf{p}_\alpha$, $\mathbf{p}_{\alpha'}$ and $\mathbf{p}_n$. Then the proper symmetrized amplitude we need to calculate is:

$$\mathcal{A} = \frac{1}{\sqrt{2}} \left(\langle \mathbf{p}_\alpha, \mathbf{p}_{\alpha'}, \mathbf{p}_n | i \rangle + \langle \mathbf{p}_{\alpha'}, \mathbf{p}_\alpha, \mathbf{p}_n | i \rangle \right) \tag{108}$$

Writing out the different Jacobi components:

$$\begin{aligned} \mathcal{A} = \frac{1}{\sqrt{2}} \Big(& \langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{\alpha' n} | i \rangle + \langle \mathbf{p}_{1=\alpha'}, \mathbf{p}_{n\alpha} | i \rangle + \langle \mathbf{p}_{1=n}, \mathbf{p}_{\alpha\alpha'} | i \rangle \\ & + \langle \mathbf{p}_{1=\alpha'}, \mathbf{p}_{\alpha n} | i \rangle + \langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{n\alpha'} | i \rangle + \langle \mathbf{p}_{1=n}, \mathbf{p}_{\alpha'\alpha} | i \rangle \Big) \end{aligned} \tag{109}$$

The first line corresponds to the three sequential contributions from the first term in the amplitude in equation 108, analogous to pane 1, 2 and 3 in figure 8. The second line are the corresponding contributions where the two alpha

particles have been interchanged (analogous to pane 4, 5 and 6 in figure 8). Now, looking at figure 10 we note that all final states accessible from ^{5}He have odd l_2 (specifically, $l_2 = 1$). Hence, the sequential amplitudes $\langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{n\alpha'}|i\rangle$ have definite symmetry under a sign change of the relative momentum $\mathbf{p}_{n\alpha'}$. Specifically:

$$\langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{n\alpha'}|i\rangle = -\langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{\alpha' n}|i\rangle \tag{110}$$

This means that the amplitude in which particle a is sent out first with momentum $\mathbf{p}_\alpha$ (corresponding to pane 1 in figure 8) is exactly canceled by the amplitude where particle b is sent out first with the same momentum (pane 5 in figure 8). The same goes for the other two ^{5}He contributions. Hence all contributions from ^{5}He drop out, leaving us with:

$$\mathcal{A} = \frac{1}{\sqrt{2}} \left(\langle \mathbf{p}_{1=n}, \mathbf{p}_{\alpha\alpha'}|i\rangle + \langle \mathbf{p}_{1=n}, \mathbf{p}_{\alpha'\alpha}|i\rangle \right) \tag{111}$$

Now, the final states accessible from ^{8}Be are all of even l_2 (see figure 11), and hence the two sequential amplitudes left are in fact equal, and so the initial symmetrization step was not necessary because the amplitude was already symmetric. We conclude:

$$\mathcal{A} = \langle \mathbf{p}_{1=n}, \mathbf{p}_{\alpha\alpha'}|i\rangle \tag{112}$$

Hence, the sequential model predicts that *the α's can not be emitted first.* This result is rather striking, especially since experiments clearly show the contributions of ^{5}He resonances in this decay (see e.g. [4] where the ^{5}He ground state is seen quite clearly, or [3] for a Dalitz plot, showing the same thing). Something went wrong - but what? Returning to equation 109 we note that the contributions that cancel each other out completely are those corresponding to two distinct physical situations - namely one of them corresponds to fragment a being ejected first and the other corresponds to fragment b being ejected first. Do these processes interfere?

To answer this question we point out that the model is built on *steady state* scattering theory. That is, the model does not work with wave-packets, but instead in plane and spherical waves, meaning that the final state fragments are *completely delocalized* - they inhibit all physical space, and time has been ignored. A proper treatment of this problem would demand a wave packet description, and we expect such a model to show that the time-difference between the two decays means that the two α-particles are too far

away from one another to interfere[12] unless they form the ^{8}Be recoil nucleus, meaning that the extreme symmetrization effect where the ^{5}He contributions completely cancel would disappear. That is, the first emitted α-particle is far away from the recoil before the second α-particle enters the scene, meaning their wave functions never overlap. Of course, if the momenta of the two α-particles are the same ($\mathbf{p}_\alpha = \mathbf{p}_{\alpha'}$) we expect to see symmetrization effects between the two cases where the same particle is emitted first - e.g. pane 1 and 4 in figure 8, but the dramatic complete cancellation happening between cases where *two different particles* are emitted first would not be expected to persist. This much more limited symmetrization effect would only lead to complete cancellation in a line in the Dalitz plot. If the axes of the Dalitz plot are chosen as the energies of the α-particles this line will be a straight line with slope 1. In conclusion, we believe that the interference between identical fragments emitted in different steps of the sequence is not treatable in our model because in reality the fragments are sufficiently localized for their wave functions to not overlap.

One way to improve the model in a quite *ad hoc* fashion, could be to eliminate the symmetric contributions that we do not expect to contribute "by hand" - i.e. we simply don't perform the symmetrization of the ^{5}He contributions. This would of course also destroy the *correct* symmetrization effect for $\mathbf{p}_\alpha = \mathbf{p}_{\alpha'}$, but it might be acceptable to knowingly make this error in such a limited region of the Dalitz plot. It would be interesting to test the model in the regime where the symmetrization can be ignored, either through simulation or by application to experimental data.

As we have just discussed, identical particle symmetrization effects between different decay paths can not contribute unless the two decays happen very close in time. Should the model not be able to treat cases where the two decays *do* happen very close in time, that is very broad (short-lived) levels in the recoil? The answer is no: any sequential model breaks down in this limit, because direct three particle effects will contribute. For example, our model does not take the Coulomb force of the first emitted particle on the other two into account. This deficiency has been pointed out for the sequential model attributed to Balamuth [5] as well, see for example [1] on page 3. Other effects such as "rescattering" where say fragment 2 scatters off of fragment 1 after the second fragmentation has happened also break the

[12]Symmetrization effects are only present when the wave functions overlap. See e.g. [11] section 7.3.

model in this limit (see [12]).

There might be an intermediate region, before the sequential model breaks down completely, but close enough to it that interference between different sequences of emission could interfere. If such an intermediate region exists, our model could suggest that certain odd l_2 decays might be suppressed there. However, nothing definitive can be concluded from this analysis, since this regime is beyond the applicability of our model.

It is a little disheartening that our model can not treat both interference between different sequential paths as well as indentical particle symmetry effects, as a detailed description combining the two effects would have been quite illuminating. We do remark however, that the model still has its merits: For instance the decay of ^{8}He resulting in an α-particle, a triton t and a neutron n would only contain effects of different sequential paths, while no identical particle symmetry effect is present to further complicate matters. Based on this analysis we expect that a model accurately describing both interference between different orders of emission and identical particle symmetrization effects must contain direct three-particle contributions.

9 Relation to Balamuth et al.

Balamuth et al. [5] studied the fragmentation of ^{12}C into three α-particles through a sequential model. They present both a formula for calculating the sequential amplitude, as well as a procedure for symmetrizing this amplitude since the final state contains three identical α-particles. First we shall see how their sequential amplitude expression is a simplified version of our main result (equation 83); then we shall examine the effect of three identical bosons in our model and compare to the symmetrization procedure outlined by Balamuth et al.

9.1 The sequential amplitude in Balamuth et al.

Balamuth et al. examine a simpler process, only going through a single level in ^{8}Be. They split up the amplitude into three components corresponding to the different Jacobi coordinates just like we do, and state the following expression for the sequential amplitude (slightly adapted to the notation used

in this thesis):

$$\mathcal{A}_1 \propto \sum_{m_{S_B}} \langle l', m_{S_A} - m_{S_B}; S_B, m_{S_B} | S_A, m_{S_A} \rangle \times Y_{l'}^{m_{S_A}-m_{S_B}}(\hat{\mathbf{p}}_1) Y_{l_2}^{m_{S_B}}(\hat{\mathbf{p}}_{23}) \times \frac{[\Omega\sqrt{2P}]_\mu \gamma_{\mu,\lambda} [\Omega\sqrt{2P}]_c \gamma_{c,\mu}}{E_0 - \gamma_{c,\mu}^2 \left[S_c(K_{23}) - S_c(K_0) + iP_c(K_{23})\right] - K_{23}} \frac{1}{\sqrt{\sqrt{K_{1B} K_{23}}}} \quad (113)$$

This expression is a special case of our main result, equation 83. Let us see why. Matching the notation, our sequential amplitude expression reads:

$$\mathcal{A}_1 = \sum_c \frac{\hbar}{\sqrt{\mu_{1,B} k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3} k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \times \Omega_c \sqrt{2P_c} \gamma_{c,\mu} \mathbf{B} \boldsymbol{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f}$$

The expression in Balamuth et al. only includes a single level in B (i.e. in ^{8}Be), and only allows for a single orbital angular momentum in both fragmentations. This means that only *one channel* is available for the first fragmentation, and only *one level* is available for the second fragmentation. Specifically, the diagonal matrices $\boldsymbol{\Omega}_\mu \sqrt{2\mathbf{P}}_\mu$ are no longer matrices but simply numbers (the sum over the projection of B is written out explicitly, contrary to our expression). Also, the level-matrix $\mathbf{B}$ is 1-dimensional (since there is only 1 level). That makes the inversion trivial (as we saw in section 7):

$$\mathbf{B} = \frac{1}{E_0 - K_{23} - \gamma_{23,B}^2 (S_{23} - B_{23} + iP_{23})} \quad (114)$$

Furthermore, only 1 final channel c is considered, removing the c-sum. Balamuth et al. also write out the angular momentum coupling explicitly instead of including it in the reduced width, like we did in section 4.5. This is the first factor in equation 113. Since all final state fragments are spin 0 the coupling is simply a Clebsch-Gordan coefficient coupling the angular momentum of

fragment 1 to the spin of the recoil. This reduces our expression to:

$$\mathcal{A}_1 = \sum_{m_{S_B}} \langle l', m_{S_A} - m_{S_B}; S_B, m_{S_B} | S_A, m_{S_A} \rangle \\ \frac{\hbar}{\sqrt{\mu_{1,R} k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3} k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \\ \times \frac{[\Omega\sqrt{2P}]_\mu \gamma_{\mu,\lambda} [\Omega\sqrt{2P}]_c \gamma_{c,\mu}}{E_0 - \gamma_{c,\mu}^2 \left[S_c(K_{23}) - S_c(K_0) + iP_c(K_{23})\right] - K_{23}} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f} \quad (115)$$

Now, Balamuth et al. don't populate the states in ^{12}C by a β-decay. They consider the rate at which it is populated to be a sort of beam intensity and simply absorb this rate in the proportionality constant. Balamuth et al. also don't investigate the effects of different levels in ^{12}C, meaning that the level matrix of ^{12}C, $\mathbf{A}$, is also absorbed in the proportionality constant. This can be done provided that the levels in ^{12}C are narrow, otherwise the proportionality constant would vary over a range of energies in ^{12}C. This is essentially the opposite situation to the one treated by Lane and Thomas, where the second fragmentation was assumed to so narrow it could be integrated away. With this, we arrive at:

$$\mathcal{A}_1 \propto \sum_{m_{S_B}} \langle l', m_{S_A} - m_{S_B}; S_B, m_{S_B} | S_A, m_{S_A} \rangle \\ \frac{\hbar}{\sqrt{\mu_{1,R} k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3} k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \\ \times \frac{[\Omega\sqrt{2P}]_\mu \gamma_{\mu,\lambda} [\Omega\sqrt{2P}]_c \gamma_{c,\mu}}{E_0 - \gamma_{c,\mu}^2 \left[S_c(K_{23}) - S_c(K_0) + iP_c(K_{23})\right] - K_{23}} \quad (116)$$

The only thing left to account for is the double square-root of the kinetic energies, appearing in the expression from Balamuth et al. These are related to the factors of $1/\sqrt{k}$ appearing in our sequential amplitude expression since:

$$k = \frac{\sqrt{2\mu K}}{\hbar} \quad (117)$$

Hence, these energy factors arise from the projection of definite angular momentum states onto states of definite linear momentum. This shows that the expression from Balamuth et al. is a special case of our sequential amplitude expression, using single-level R-matrix theory and considering spin-0 final state fragments.

9.2 Triple-α symmetrization

Let us consider how symmetrization of three α-particles in the final state (the case of Balamuth et al.) would be treated in our sequential model. Here we off course have *three* identical particles, not two like we had in section 8, but they are all spin-0 bosons, similar to our discussion there, reducing the problem to only including spatial parts. To ensure proper symmetry under exchange of any two α's the standard approach would be to use a final state consisting of all six permutations:

$$\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma| + \langle \mathbf{p}_\beta, \mathbf{p}_\gamma, \mathbf{p}_\alpha| + \langle \mathbf{p}_\gamma, \mathbf{p}_\alpha, \mathbf{p}_\beta| + \langle \mathbf{p}_\alpha, \mathbf{p}_\gamma, \mathbf{p}_\beta| + \langle \mathbf{p}_\beta, \mathbf{p}_\alpha, \mathbf{p}_\gamma| + \langle \mathbf{p}_\gamma, \mathbf{p}_\beta, \mathbf{p}_\alpha| \tag{118}$$

This results in 6 sets of Jacobi coordinates. First we note that any cyclic permutation of the labels of the α's results in the exact same sequential amplitudes, since they result in the same set of Jacobi coordinates. This allows us to reduce our final state to a sum of one of the cyclic and one of the acyclic permutations:

$$\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma| + \langle \mathbf{p}_\alpha, \mathbf{p}_\gamma, \mathbf{p}_\beta| \tag{119}$$

As we have seen, a sum like this corresponds to six Jacobi coordinates, related to each other in pairs by a sign change of the relative momentum $\mathbf{p}_{23}$. Since only even values of l_2 are allowed (see figure 7) we know that these sequential amplitudes are in fact *identical*, meaning that the amplitude:

$$\langle \mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_\gamma|i\rangle = \langle \mathbf{p}_{1=\alpha}, \mathbf{p}_{\beta\gamma}|i\rangle + \langle \mathbf{p}_{1=\beta}, \mathbf{p}_{\gamma\alpha}|i\rangle + \langle \mathbf{p}_{1=\gamma}, \mathbf{p}_{\alpha\beta}|i\rangle \tag{120}$$

is in fact symmetric under exchange of any two of the α's. Our model predicts that *if we include all three sequences of decays, our amplitude is automatically symmetric under interchange of any two fragments*, in this particular case.

Balamuth et al. perform this sum over the three distinct sequences and calls it "order of emission interference effects" at the bottom of page 9, but refers to it as boson symmetrization later in the same paragraph. Though it is in fact both (as we have just shown), the reasoning is not very precise and we must emphasize that generally, the interference effects between different orders of emission (corresponding to decomposing the amplitude into the three Jacobi-coordinates) *is not the same* as interchange symmetrization due to identical particles, as we clearly saw in the example on ^{9}Li. They just so happen to be solved by the same trick in the case of the particularly simple triple-α case. As mentioned earlier, sequential models might not be

particularly well suited to treating identical particle symmetrization effects generally, and we believe a more realistic three-body treatment might be necessary to fully capture both effects of symmetrization and interference between different sequential decay paths.

10 Conclusion

In this thesis we have investigated three-body decays. Specifically we have tried modeling three-body decays as sequences of two-body decays, in the hope that the sequential treatment could capture much of the relevant physics of these interesting processes, without dealing with the complicated three-body physics of the direct three-body decays omitted in this treatment. A model was presented, and an expression for a sequential amplitude $\mathcal{A}_1$ was derived in section 5. The expression for the sequential amplitude (equation 83) was obtained by combining two R-matrix expressions in sequence. The result was:

$$\mathcal{A}_1 = \sum_c \frac{\hbar}{\sqrt{\mu_{1,B} k_1}} Y_{l'}^{m_{l'}}(\hat{\mathbf{p}}_1) \frac{\hbar}{\sqrt{\mu_{2,3} k_{23}}} Y_{l_2}^{m_{l_2}}(\hat{\mathbf{p}}_{23}) \times \Omega_c \sqrt{2P_c} \gamma_{c,\mu} \mathbf{B}\boldsymbol{\Omega}_\mu \sqrt{\mathbf{2P}}_\mu \gamma_{\mu,\lambda} \mathbf{A} \vec{g}_{\lambda,i} \sqrt{f}$$

An example was included in section 6, to concretize the model, and in section 7 the sequential amplitude expression was shown to reproduce the result in the R-matrix review by Lane & Thomas, [8] chapter XIII in the single level limit they consider.

The consequences of identical particle symmetrization was then investigated in chapter 8. First a general example of two identical spin-0 bosons was included, then a more concrete application of the model to the case of ^{9}Li was presented. The prediction of the model was in strict discordance with experiments, and we hypothesized that the sequential nature of the model might make symmetrization between *different* sequential decay paths problematic.

Lastly, our model was compared to the work of Balamuth et al. on the triple-α decay of ^{12}C. We showed that our sequential amplitude expression can reproduce the formula by Balamuth et al. and we point out that Balamuth et al. are not very precise in their article [5], on what effects arise

from summing different sequential decay paths, and what effects arise from identical particle symmetrization. We show that in the case of the triple-α process investigated by Balamuth et al. these two effects are indistinguishable (according to our model).

The model developed in this thesis could be useful in studying sequential three-body decays when no identical particles are present, or when symmetrization effects between distinct sequential decay paths can be ignored. We suggest the decay of ^{8}He as a candidate for a decay with no identical particles.

11 References

[1] H. O. U. Fynbo et al. "Clarification of the Three-Body Decay of ^{12}C (12.71 MeV)". In: *Physical Review Letters* 91.8 (2003).

[2] H. O. U. Fynbo et al. "Revised rates for the stellar triple-α process from measurement of ^{12}C nuclear resonances". In: *Nature* 433 (2005), pp. 136–139.

[3] Y. Prezado et al. "Large asymmetry in the strongest β-transition for $A = 9$". In: *Physics Letters B* 576 (2003), pp. 55–61.

[4] Y. Prezado et al. "Low-lying resonance states in the ^{9}Be continuum". In: *Physics Letters B* 618 (2005), pp. 43–50.

[5] D. P. Balamuth, R. W. Zurmühle, and S. L. Tabor. "Isospin-forbidden alpha decay of the 15.11 MeV state in ^{12}C". In: *Physical Review C, Vol. 10, No. 3* (1974).

[6] Solveig Hyldegaard. "Beta-decay studies of ^{8}Be and ^{12}C". PhD thesis. Department of Physics and Astronomy, Aarhus University, 2010.

[7] John David Jackson. *Kinematics*. URL: `https://pdg.lbl.gov/2023/web/viewer.html?file=../reviews/rpp2023-rev-kinematics.pdf`. (accessed: 10.04.2024).

[8] A. M. Lane and R. G. Thomas. "R-Matrix Theory of Nuclear Reactions". In: *Reviews of Modern Physics, Vol. 30, No. 2, Part 1* (1958).

[9] E. Nielsen et al. "The three-body problem with short-range interactions". In: *Physics Reports* 347 (2001), pp. 373–459.

[10] M. A. Preston. *Physics of the Atomic Nucleus*. Addison-Wesley, 1962.

[11] Jun John Sakurai. *Modern Quantum Mechanics*. Cambridge University Press, 2021.

[12] Erich W. Schmid and Horst Ziegelmann. *The Quantum Mechanical Three-Body Problem*. 1st edition. Pergamon Press, 1974.

[13] E. P. Wigner and L. Eisenbud. "Higher angular momenta and long range interaction in resonance reactions". In: *Physical Review* 72.1 ().

[14] Wikipedia contributors. *Racah W-coefficient — Wikipedia, The Free Encyclopedia*. [Online; accessed 15-October-2024]. 2023. URL: `https://en.wikipedia.org/w/index.php?title=Racah_W-coefficient&oldid=1186226163`.